

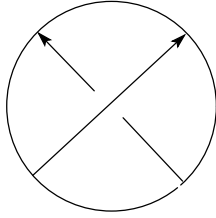
LOCAL-MOVE-IDENTITIES FOR THE $\mathbb{Z}[t, t^{-1}]$ -ALEXANDER POLYNOMIALS OF 2-LINKS, THE ALINKING NUMBER, AND HIGH DIMENSIONAL ANALOGUES

EIJI OGASA

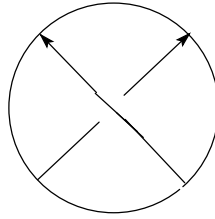
ABSTRACT. A well-known identity $\hat{\Delta}_{L_+} - \hat{\Delta}_{L_-} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \hat{\Delta}_{L_0}$ holds for three 1-links L_+ , L_- , and L_0 which satisfy a famous local-move-relation, where $\hat{\Delta}_L$ becomes the Alexander-Conway polynomial of L if we let $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. We prove a new local-move-identity for the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomials of 2-links, which is a 2-dimensional analogue of the 1-dimensional one. In the 1-dimensional link case there is a well-known relation between the Alexander-Conway polynomial and the linking number. As its 2-dimensional analogue, we find a relation between the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomials of 2-links and the alinking number of 2-links. We show high dimensional analogues of these results. Furthermore we prove that in the 2-dimensional case we cannot normalize the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomials to be compatible with our identity but that in a high-dimensional case we can do that to be compatible with our new identity.

1. INTRODUCTION AND MAIN RESULTS

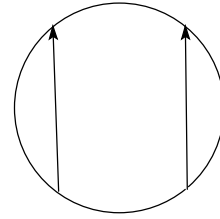
Suppose that three 1-dimensional links L_+ , L_- , and $L_0 \subset S^4$ differ only in a 3-ball B as shown below.



L_+



L_-



L_0

It is very well-known that then we have the identity (*)

$$\hat{\Delta}_{L_+} - \hat{\Delta}_{L_-} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \hat{\Delta}_{L_0},$$

where $\hat{\Delta}_L$ denotes the normalized Alexander polynomial of L . (If we let $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, $\hat{\Delta}_L$ becomes the Alexander-Conway polynomial.) It is also very well-known that the Jones

polynomial satisfies a similar local-move-identity and that there are several relations between local-moves on 1-links and their invariants. (See [2, 3, 8, 9, 10] etc.)

In [21, Theorem 4.1] we showed a 2-link version of the identity (*). We cite it after we state its corollary and an example. We strength it and obtain a main result of this paper, Theorem 4.4, cited in several paragraphs.

The corollary is as follows: Suppose that two 2-dimensional spherical knots, K_+ and K_- , $\subset S^4$ and a submanifold $K_0 \subset S^4$ differ only in a 4-ball B trivially embedded in S^4 as shown in Figures 2.1-2.3 in §2. (This ordered set (K_+, K_-, K_0) is called a (1,2)-pass-move-triple. An example is made from Figure 1.1, 1.3, and 1.5, which are explained in the following paragraph. See §2 for the precise definition.) Then we have the following (#): there is a polynomial $\Delta_{\nu, K_*}(t)$ ($*$ = +, -, 0 and $\nu = 1, 2$) which represents the $\mathbb{Q}[t, t^{-1}]$ - ν -Alexander polynomial for K_* , and we have the identity $\Delta_{\nu, L_+}(t) - \Delta_{\nu, L_-}(t) = (t - 1) \cdot \Delta_{\nu, L_0}(t)$.

K_+ in Figure 1.1, K_- in Figure 1.3, and K_0 in Figure 1.5 are constructed as follows: Embed $F = (S^1 \times S^2)$ -open B^3 in S^4 . The boundary of F in S^4 is a 2-knot. Let it be a trivial 2-knot K_+ as drawn in Figure 1.1. Carry out a ‘local-move’ on the 2-knot K_+ in a 4-ball, which is denoted by a dotted circle in Figure 1.2. This local-move is called the (1,2)-pass-move (see §2 for the precise definition). Note that the above operation is done only in the 4-ball. The (1,2)-pass-move changes the trivial 2-knot in Figure 1.1 (resp. 1.2) into a 2-knot in Figure 1.3. We can prove that the knot in Figure 1.3 is nontrivial by using Seifert matrices and the Alexander polynomial. We use the fact that S^1 and S^2 can be ‘linked’ in S^4 . Note that S^1 and S^2 are included in F as shown in Figure 1.4. K_0 is drawn in Figure 1.5. If we give appropriate orientations, we can let $\Delta_{K_+} = t$, $\Delta_{K_-} = 2t - 1$, and $\Delta_{K_0} = -1$. Hence $\Delta_{K_+} - \Delta_{K_-} = (t - 1) \cdot \Delta_{K_0}$ holds.

[21, Theorem 4.1] is as follows: Let $L_+ = (L_{+,1}, \dots, L_{+,m_+})$ be a 2-dimensional closed oriented submanifold $\subset S^4$. Let each $L_{+,i}$ be connected. Let $g_{+,i}$ be the genus of $L_{+,i}$. Let $m_+ = 1 + \sum_1^{m_+} g_{+,i}$. Let (L_+, L_-, L_0) be a (1,2)-pass-move-triple. Then we have the above (#), where we replace K_* with L_* .

In [21, Proposition 4.3] we proved that we cannot normalize the Alexander polynomials to be compatible with this local-move-identity.

In [11, 12, 13, 17, 19, 20, 21, 22, 23], furthermore, we proved several relations between local-moves on n -knots and their invariants ($n \in \mathbb{N}$).

We state one of our main results, Theorem 4.4, whose $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial case is [21, Theorem 4.1]. The former is stronger than the latter because the former does not follow from the latter directly (see Propositions 4.3 and 4.5). Furthermore we prove a high dimensional analogue of this main theorem (see Theorem 6.3 quoted in this section.)

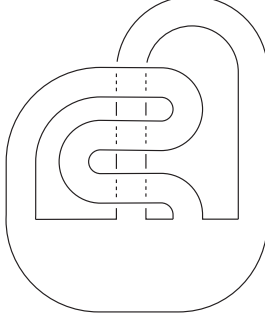


Figure 1.1: **A trivial 2-knot K_+ and F . $\partial F = K_+$.**

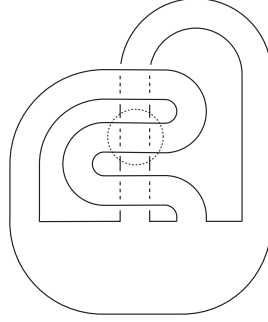


Figure 1.2: **A local-move will be carried out in the dotted 4-ball. The resulting 2-knot is a nontrivial 2-knot K_- .**

Theorem 4.4. *Let $L_+ = (L_{+,1}, \dots, L_{+,m_+})$ be a 2-dimensional closed oriented submanifold $\subset S^4$. Let each $L_{+,i}$ be connected. Let $g_{+,i}$ be the genus of $L_{+,i}$. Let $m_+ = 1 + \sum_1^{m_+} g_{+,i}$. Let (L_+, L_-, L_0) be a $(1, 2)$ -pass-move-triple. Then there is a polynomial $\Delta_{\nu, K_*}(t) (* = +, -, 0 \text{ and } \nu = 1, 2)$ which represents the $\mathbb{Z}[t, t^{-1}]$ - ν -Alexander polynomial for K_* , and we have the identity*

$$\Delta_{\nu, L_+}(t) - \Delta_{\nu, L_-}(t) = (t - 1) \cdot \Delta_{\nu, L_0}(t).$$

K_+ in Figure 1.1, K_- in Figure 1.3, and K_0 in Figure 1.5 make not only an example of [21, Theorem 4.1] but also Theorem 4.4.

Theorem 6.3 is as follows. The terms and definitions needed for it are in the body of the paper. In the 2-dimensional case we cannot normalize the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial to be compatible with the local-move-identity as we explained a few paragraphs before, but in a case of $(4k + 1)$ -dimensional case we can define the ‘normalized’ Alexander

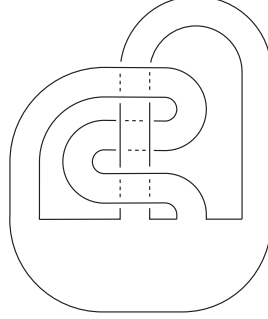


Figure 1.3: **A nontrivial 2-knot K_-**

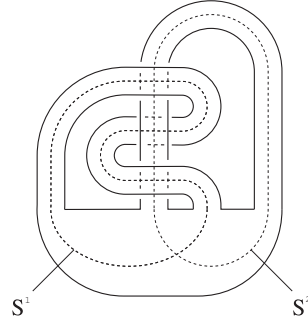


Figure 1.4: S^1 and S^2 in F whose boundary is the 2-knot

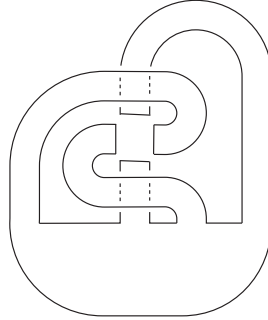


Figure 1.5: K_0 is a trivial 2-knot in this case.

polynomial (see Definition 6.1 and Theorem 6.2) associated with a local-move defined in §5.

Theorem 6.3. *Let K_+ be a $(4k + 1)$ -knot $\subset S^{4k+3}$. Let (K_+, K_-, K_0) be a twist-move-triple. Then*

$$\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \hat{\Delta}_{K_0}(t),$$

where $\hat{\Delta}_K(t)$ denotes the normalized Alexander polynomial of K .

In the 1-dimensional case we have the following fact ([6]): Let L be a 2-component 1-link. Let $\hat{\Delta}_L(t)$ be the normalized Alexander polynomial of L . Then $\left. \frac{\hat{\Delta}_K(t)}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \right|_{t=1}$ is the linking number of L . Let K_+, K_- , and K_0 be as in the first paragraph of this section. Let K_+ be a 1-knot. Then K_0 is a 2-component 1-link and $\left. \frac{\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t)}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2} \right|_{t=1}$ is the linking number of K_0 .

In §4 we prove a 2-dimensional analogue of this result: we show a relation between the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of 2-dimensional closed oriented submanifolds $\subset S^4$ and the alinking number (Theorems 4.8 and 4.13). In §6 furthermore we prove some high dimensional analogues (Theorem 6.7 and Corollary 6.8). We cite the above theorems here. The terms and definitions needed for them are in the body of the paper.

Theorem 4.8. *Let $L = (L_1, L_2)$ be an (S^2, T^2) -link $\subset S^4$. Let $\Delta_{1,L}(t)$ be a polynomial which represents the $\mathbb{Z}[t, t^{-1}]$ -1-Alexander polynomial for L . Then*

$$\left| \left. \frac{\Delta_{1,L}(t)}{(t-1)} \right|_{t=1} \right|$$

is the pseudo-alinking number of L , where $| \cdot |$ denotes the absolute value.

Theorem 4.13. *Let $L = (K_1, K_2)$ be a ribbon (S^2, T^2) -link. Then the alinking number of L is $\left| \left. \frac{\Delta_{1,L}(t)}{(t-1)} \right|_{t=1} \right|$.*

Theorem 6.7. *Let K be a $(4k+1)$ -dimensional closed oriented submanifold $\subset S^{4k+3}$ whose homotopy type is $S^{2k} \times S^{2k+1}$. Let $\hat{\Delta}_K(t)$ be the normalized Alexander polynomial of K . Then the pseudo-twinkling number of K is*

$$\left. \frac{\hat{\Delta}_K(t)}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \right|_{t=1}$$

Corollary 6.8. *Let K_+ be a $(4k+1)$ -knot $\subset S^{4k+3}$. Let (K_+, K_-, K_0) be a twist-move-triple. Then the pseudo-twinkling number of K_0 is*

$$\left. \frac{\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t)}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2} \right|_{t=1}$$

Table of Contents

- §1 Introduction and main results
- §2 Review of (1,2)-pass-moves on 2-knots
- §3 Review of the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial for (not necessarily connected)
 n -dimensional closed oriented submanifolds in S^{n+2} ($n \geq 2$)
- §4 Main theorems in the 2-dimensional case
- §5 Review of twist-moves on high dimensional knots
- §6 Main theorems in the $4k+1$ dimensional case
- §7 Proof of results of §4
- §8 Proof of results of §6
- §9 A problem

2. REVIEW OF (1,2)-PASS-MOVES ON 2-KNOTS

The local-move which associates the identity $(*)$ in the first paragraph in §1 is very easy as drawn there. In high dimensional case we must begin by explaining what kind of local-moves we use. We review the (1,2)-pass-move on 2-dimensional closed oriented submanifolds $\subset S^4$, which are defined in [19].

We work in the smooth category. A (not necessarily connected) 2-dimensional smooth, closed oriented submanifold $L \subset S^4$ is called an m -component 2-(dimensional) (spherical) link if L consists of m connected components and each connected component is a 2-sphere. If L is 1-component 2-link, then L is called a (spherical) 2-knot. We say that (not necessarily connected) 2-dimensional smooth, closed, oriented submanifolds L_1 and $L_2 \subset S^4$ are *equivalent* if there exists an orientation preserving diffeomorphism $f : S^4 \rightarrow S^4$ such that $f(L_1) = L_2$ and that $f|_{L_1} : L_1 \rightarrow L_2$ can be regarded as an order and orientation preserving diffeomorphism map.

The (1,2)-pass-move is a local-move. Here, ‘local-move’ means that when we make K_+ into K_- (resp. K_- into K_0 , K_0 into K_-) vice versa in Definition 2.1, we make a change only in B and that we do not any requirement on diffeomorphism type or homeomorphism type of K_+ (resp. K_- , K_0) other than the change only in B .

Definition 2.1. Let L_+ , L_- , and L_0 be (not necessarily connected) 2-dimensional closed oriented submanifolds $\subset S^4$. We say that (L_+, L_-, L_0) is a (1,2)-pass-move-triple if L_+ , L_- , and L_0 differ only in a 4-ball B trivially embedded in S^4 with the following properties: $B \cap L_+$ is drawn as in Figure 2.1. $B \cap L_-$ is drawn as in Figure 2.2. $B \cap L_0$ is drawn as in Figure 2.3. Note that we do not assume how many connected components of L_+ intersect B . Furthermore we say that L_+ (resp. L_-) is obtained from L_- (resp. L_+) by one (1,2)-pass-move. If L is equivalent to L' and if L' is obtained from L'' by a sequence of (1,2)-pass-moves, we say that L is (1,2)-pass-move-equivalent to L'' .

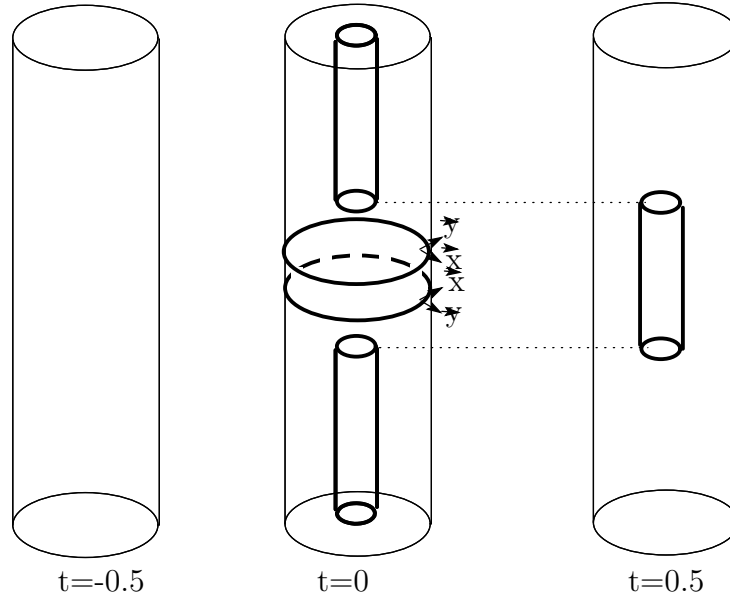


Figure 2.1: L_+ of a $(1, 2)$ -pass-move-triple

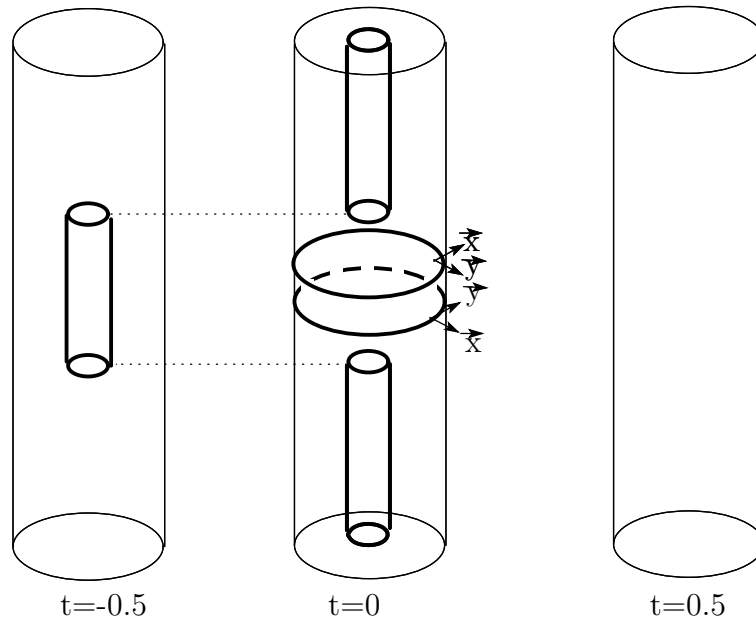


Figure 2.2: L_- of a $(1, 2)$ -pass-move-triple

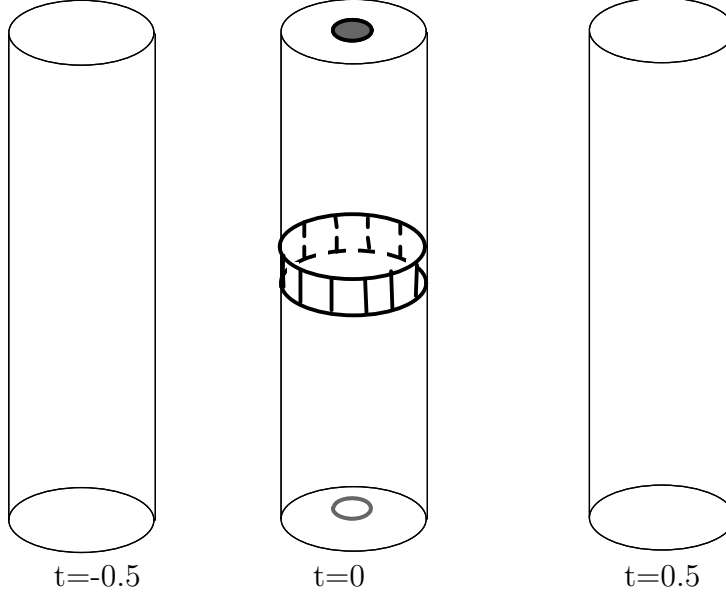


Figure 2.3: L_0 of a $(1, 2)$ -pass-move-triple

We regard B as $(\text{a close 2-disc } P) \times [0, 1] \times \{t \mid -1 \leq t \leq 1\}$. Let $B_t = (\text{the close 2-disc } P) \times [0, 1] \times \{t\}$. Then $B = \cup B_t$. In Figures 2.1, 2.2 and 2.3, we draw $B_{-0.5}, B_0, B_{0.5} \subset B$. We draw L_+, L_- , and L_0 by the bold line. The fine line denotes ∂B_t .

$B \cap L_+$ (resp. $B \cap L_-$) is diffeomorphic to $D^2 \amalg D^2 \amalg (S^1 \times [0, 1])$, where \amalg denotes the disjoint union. $B \cap L_+$ has the following properties: $B_t \cap L_+$ is empty for $-1 \leq t < 0$ and $0.5 < t \leq 1$. $B_0 \cap L_+$ is $D^2 \times \{0.4\} \amalg D^2 \times \{0.6\} \amalg (S^1 \times [0, 0.3]) \amalg (S^1 \times [0.7, 1])$. $B_{0.5} \cap L_+$ is $S^1 \times [0.3, 0.7]$. $B_t \cap L_+$ is diffeomorphic to $S^1 \amalg S^1$ for $0 < t < 0.5$. (Here we draw $S^1 \times [0, 1]$ to have the corner in B_0 and in $B_{0.5}$. Strictly to say, $B \cap L_+$ in B is a smooth embedding which is obtained by making the corner smooth naturally.)

$B \cap L_-$ has the following properties: $B_t \cap L_-$ is empty for $-1 \leq t < -0.5$ and $0 < t \leq 1$. $B_0 \cap L_-$ is $D^2 \times \{0.4\} \amalg D^2 \times \{0.6\} \amalg (S^1 \times [0, 0.3]) \amalg (S^1 \times [0.7, 1])$. $B_{-0.5} \cap L_-$ is $S^1 \times [0.3, 0.7]$. $B_t \cap L_-$ is diffeomorphic to $S^1 \amalg S^1$ for $-0.5 < t < 0$.

In Figure 2.1 (resp. 2.2) there are an oriented cylinder $S^1 \times [0, 1]$ and two oriented discs D^2 . We do not make any assumption about the orientation of the cylinder. We suppose that each arrow \vec{x}, \vec{y} in Figure 2.1 (resp. 2.2) is a tangent vector of each disc at a point. (Note we use the same notations \vec{x} (resp. \vec{y}) for different arrows.) The orientation of each disc in Figure 2.1 (resp. 2.2) is determined by the each set $\{\vec{x}, \vec{y}\}$. The orientation of $B \cap L_+$ (resp. $B \cap L_-$) coincides with that of the cylinder and that

of the disc. We can suppose that there is a Seifert hypersurface V such that $V \cap B$ is $P \times [0.3, 0.7]$.

$B \cap L_0$ is a disjoint union of two 2-discs and an annulus as drawn in Figure 2.3. One of the 2-discs is in $(\text{the close 2-disc } P) \times \{0\} \times \{0\}$ and the other in $(\text{the close 2-disc } P) \times \{1\} \times \{0\}$. The annulus is in $(\partial(\text{the close 2-disc } P)) \times [0.4, 0.6] \times \{0\}$.

Recall that an example of (1,2)-pass-move-triples is drawn in §1.

Note 2.2. In the (1,2)-pass-move case we have the following examples: Let V be a Seifert hypersurface for a 2-knot K . Suppose that V is diffeomorphic to $((S_a^1 \times S_a^2) \# (S_b^1 \times S_b^2)) - \text{open } B^3$. Take orientations of $S_i^1 \times *$ and $* \times S_j^2$ ($i, j \in \{a, b\}$) so that the intersection product of $S_i^1 \times *$ and $* \times S_j^2$ is δ_{ij} . Suppose that the Seifert pairing of $S_a^1 \times *$ and $* \times S_b^2$ is one. If we change the orientations of $S_b^1 \times *$ and $* \times S_b^2$, then the intersection product of $S_i^1 \times *$ and $* \times S_j^2$ does not change but the Seifert pairing of $S_a^1 \times *$ and $* \times S_b^2$ changes $+1$ into -1 .

It means the following: Suppose that we know that one of 2-dimensional closed oriented submanifolds, K and J , $\subset S^4$ is K_+ , and that the other K_- . Then we cannot distinguish K_+ from K_- without the information of the orientation how K_+ and K_- intersect B .

On the other hand, in the case of the twist-move on the $(4k+1)$ -dimensional submanifolds, we can distinguish K_+ from K_- . See Note 5.1.

In [19] we introduce the ribbon-move for closed oriented 2-dimensional submanifolds $\subset S^4$. The ribbon-move is much connected with the (1,2)-pass-move (see [19, Proposition 4.2]). If we replace ‘(1,2)-pass-move’ (resp. ‘(1,2)-pass-move-triple’) with ‘ribbon-move’ (resp. ‘ribbon-move-triple’) in the theorems of this paper, similar theorems could hold. We draw (a part of a figure of) a ribbon-move-triple in Figure 2.4-2.6. See [19] for the precise definition.

3. REVIEW OF THE $\mathbb{Q}[t, t^{-1}]$ -ALEXANDER POLYNOMIAL FOR (NOT NECESSARILY CONNECTED) n -DIMENSIONAL CLOSED ORIENTED SUBMANIFOLDS IN S^{n+2} ($n \geq 2$)

In this section we review the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial for (not necessarily connected) n -dimensional closed oriented submanifolds $\subset S^{n+2}$ ($n \geq 2$). In §4 we define the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial for 2-dimensional closed oriented submanifolds $\subset S^4$. In §6 we define the ‘normalized’ Alexander polynomial for a kind of $(4k+1)$ -submanifolds $\subset S^{4k+3}$. Of course these invariants are connected each other.

Let $K = (K_1, \dots, K_\xi)$ be an n -dimensional closed oriented submanifold of S^{n+2} ($n \in \mathbb{N}$). Let each K_i be connected. If K_i is PL homeomorphic to the standard sphere, K_i is called an n -dimensional (spherical) knot. If each K_i is an n -knot, K is called an ξ -component n -dimensional (spherical) link.

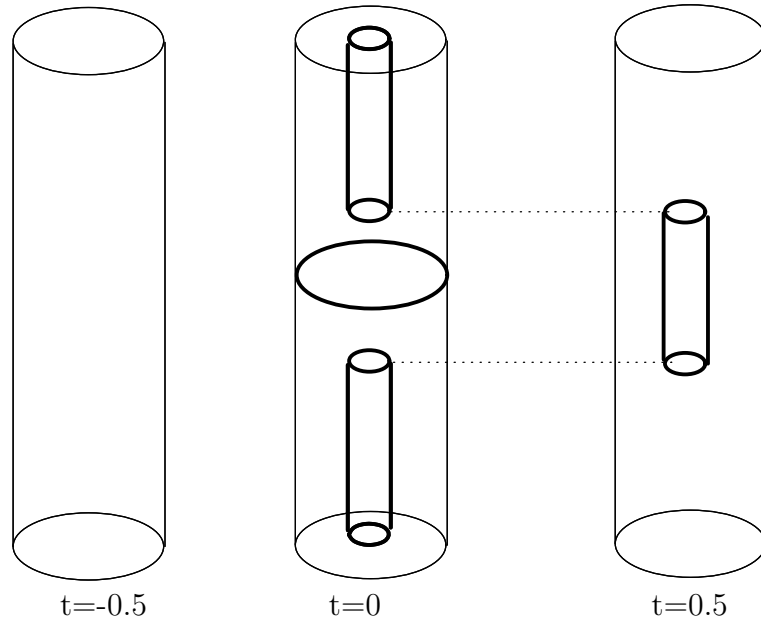


Figure 2.4: L_+ of a ribbon-move-triple.

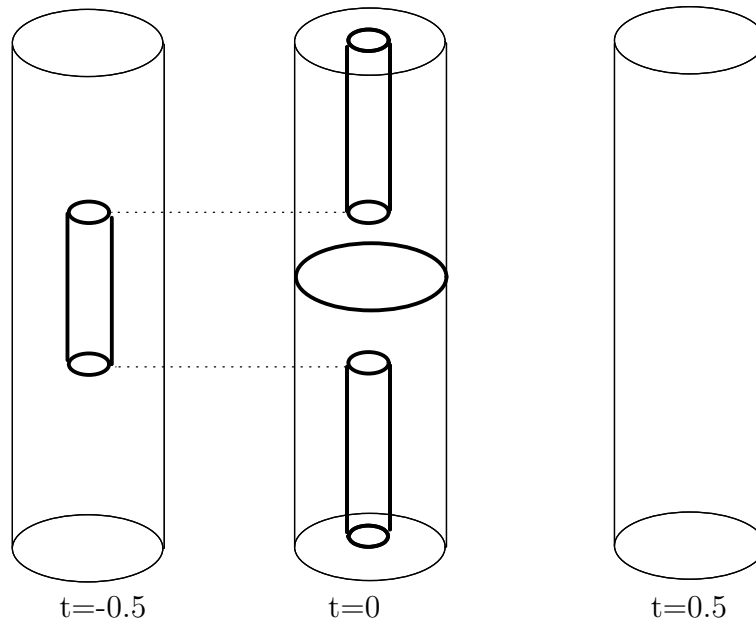


Figure 2.5: L_- of a ribbon-move-triple.

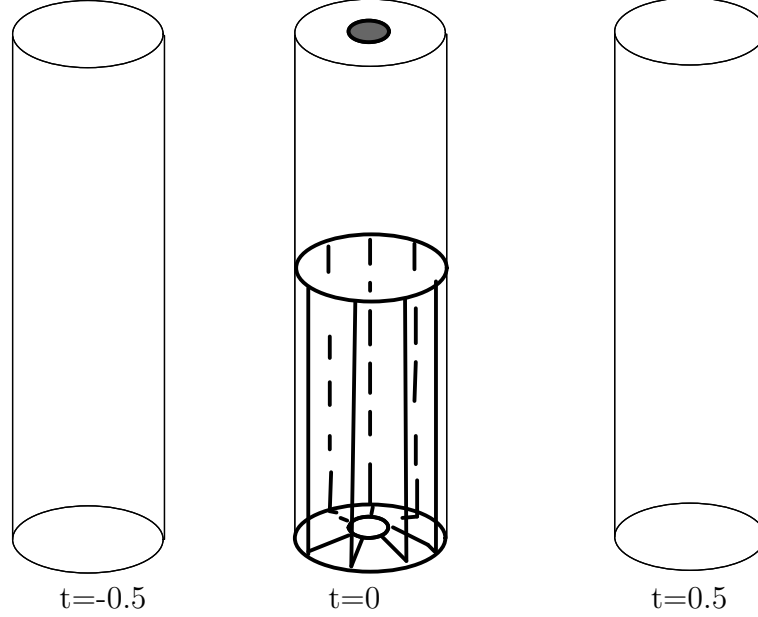


Figure 2.6: L_0 of a ribbon-move-triple.

It is known that the tubular neighborhood of K is diffeomorphic to $K \times D^2$ (see [14, pages 49 and 50]). Let $X = \overline{S^{n+2} - (K \times D^2)}$. By using the orientation of S^{n+2} and that of K , we can determine an orientation of ∂D^2 . Take a homomorphism $\alpha : H_1(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ to carry all $[\partial D^2]$ with the orientations to $+1$. Take the infinite cyclic covering $\pi : \tilde{X} \rightarrow X$ associated with α . \tilde{X} is called the *canonical cyclic covering space* of K . We can regard $H_p(\tilde{X}; \mathbb{Z})$ (resp. $H_p(\tilde{X}; \mathbb{Q})$) as a $\mathbb{Z}[t, t^{-1}]$ -module (resp. $\mathbb{Q}[t, t^{-1}]$ -module) by using the covering translation $\tilde{X} \rightarrow \tilde{X}$ defined by α . It is called the $\mathbb{Z}[t, t^{-1}]$ -*p-Alexander module* (resp. $\mathbb{Q}[t, t^{-1}]$ -*p-Alexander module*).

Definition 3.1. According to module theory, it holds that any $\mathbb{Q}[t, t^{-1}]$ -module is congruent to

$$(\mathbb{Q}[t, t^{-1}]/\lambda_1) \oplus \cdots \oplus (\mathbb{Q}[t, t^{-1}]/\lambda_l) \oplus (\oplus^k \mathbb{Q}[t, t^{-1}]),$$

where we have the following:

- (1) $\lambda_* \in \mathbb{Q}[t, t^{-1}]$ is not zero,
- (2) λ_* is not the $\mathbb{Q}[t, t^{-1}]$ -balanced class of 1,
- (3) k is the rank of the free part.

Two polynomials, $f(t)$ and $g(t)$, $\in \mathbb{Q}[t, t^{-1}]$ are said to be $\mathbb{Q}[t, t^{-1}]$ -balanced if there is an integer n and a nonzero rational number r such that $f(t) = r \cdot t^n \cdot g(t)$.

Let $H_p(\tilde{X}; \mathbb{Q})$ be as above. Then the $\mathbb{Q}[t, t^{-1}]$ - p -Alexander polynomial is the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\begin{cases} \text{the product } \lambda_1 \cdot \dots \cdot \lambda_l & \text{if } k = 0 \text{ and } H_p(\tilde{X}; \mathbb{Q}) \not\cong 0, \\ 0 & \text{if } k \neq 0, \\ 1 & \text{if } H_p(\tilde{X}; \mathbb{Q}) \cong 0. \end{cases}$

In this paper manifolds (resp. submanifolds) include manifolds-with-boundary (resp. submanifolds-with-boundary). A *Seifert hypersurface* for an n -dimensional oriented closed submanifold K in S^{n+2} is an $(n+1)$ -dimensional oriented connected compact submanifold in S^{n+2} whose boundary is K ($n \in \mathbb{N}$). Note that Seifert hypersurfaces exist by obstruction theory (see [14, pages 49 and 50]). Note that there are two cases that K is not connected and that K is connected.

Let V be a Seifert hypersurface for the above n -submanifold K . Let x_1, \dots, x_μ be p -cycles in V which compose a basis of $H_p(V; \mathbb{Z})/\text{Tor}$. Let y_1, \dots, y_ν be $(n+1-p)$ -cycles in V which compose a basis of $H_{n+1-p}(V; \mathbb{Z})/\text{Tor}$. Push y_i into the positive (resp. negative) direction of the normal bundle of V . Call it y_i^+ (resp. y_i^-). A $(p, n+1-p)$ -positive Seifert matrix for the above submanifold K associated with V represented by an ordered basis, $\{x_1, \dots, x_\mu\}$, and an ordered basis, $\{y_1, \dots, y_\nu\}$, is a $(\mu \times \nu)$ -matrix

$$S = (s_{ij}) = (\text{lk}(x_i, y_j^+)).$$

A $(p, n+1-p)$ -negative Seifert matrix for the above submanifold K associated with V represented by an ordered basis, $\{x_1, \dots, x_\mu\}$, and an ordered basis, $\{y_1, \dots, y_\nu\}$, is a matrix

$$N = (n_{ij}) = (\text{lk}(x_i, y_j^-)).$$

We have the following: Let S and N be as above. Then $S - N$ represents the map $\{H_p(V; \mathbb{Z})/\text{Tor}\} \times \{H_{n+1-p}(V; \mathbb{Z})/\text{Tor}\} \rightarrow \mathbb{Z}$ which is defined by the intersection product. We call $t \cdot S - N$ the $(p, n+1-p)$ -Alexander matrix for K associated with V represented by an ordered basis, $\{x_1, \dots, x_\mu\}$, and an ordered basis, $\{y_1, \dots, y_\nu\}$. S and N (resp. S and $t \cdot S - N$, N and $t \cdot S - N$) are said to be *related* if S and N (resp. S and $t \cdot S - N$, N and $t \cdot S - N$) are defined by using the same V , the same $\{x_1, \dots, x_\mu\}$, and the same $\{y_1, \dots, y_\nu\}$. We sometimes abbreviate $(p, n+1-p)$ -positive Seifert matrix (resp. $(p, n+1-p)$ -negative Seifert matrix, $(p, n+1-p)$ -Alexander matrix) to p -Seifert matrix (resp. p -negative Seifert matrix, p -Alexander matrix) when it is clear from the context.

Proposition 3.2. *Let K be an n -dimensional oriented closed submanifold $\subset S^{n+2}$. Let S_p (resp. N_p) be a p -positive (resp. negative) Seifert matrix for K associated with V represented by an ordered basis, $\{x_1, \dots, x_\mu\}$, and an ordered basis, $\{y_1, \dots, y_\nu\}$. Suppose*

$$\mu = \nu.$$

Suppose that the homomorphism on $H_{p-1}(\Pi_\infty^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{p-1}(\Pi_\infty^\infty Y; \mathbb{Q})$ defined by a $(p-1)$ -Alexander matrix is injective. Then the p - $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial is

the $\mathbb{Q}[t, t^{-1}]$ -balanced class of ‘the determinant of p -Alexander matrix’

$$\det(t \cdot S_p - N_p).$$

Note. Of course $\mu \neq \nu$ in general.

Proof of Proposition 3.2. Take the above $X = \overline{S^{n+2} - (K \times D^2)}$, \tilde{X} , V . Let $V \times [-1, 1]$ be the tubular neighborhood of V in X . Let $Y = X - V$. Consider the Mayer-Vietoris exact sequence:

$$H_{\natural}(\Pi_{-\infty}^{\infty} V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{\natural}(\Pi_{-\infty}^{\infty} Y; \mathbb{Q}) \rightarrow H_{\natural}(\tilde{X}; \mathbb{Q}),$$

where $\Pi_{-\infty}^{\infty} V \times [-1, 1]$ is the lift of $V \times [-1, 1]$, and where $\Pi_{-\infty}^{\infty} Y$ is the lift of Y .

This completes the proof of Proposition 3.2. \square

Proposition 3.3. Let N_p be a $(p, n+1-p)$ -negative Seifert matrix for K associated with V represented by an ordered basis, $\{x_1, \dots, x_{\mu}\}$, and an ordered basis, $\{y_1, \dots, y_{\nu}\}$. Let S_{n+1-p} be a $(n+1-p, p)$ -positive Seifert matrix for K associated with V represented by an ordered basis, $\{y_1, \dots, y_{\nu}\}$, and an ordered basis, $\{x_1, \dots, x_{\mu}\}$. Then we have

$$N_p = (-1)^{p \cdot n+1} \cdot S_{n+1-p}.$$

Proof of Proposition 3.3. By the definition of x_i^+ and y_i^- , $\text{lk}(y_i, x_j^+) = \text{lk}(y_i^-, x_j)$. By [15, page 541], $\text{lk}(y_i^-, x_j) = (-1)^{p(n+1-p)+1} \text{lk}(x_j, y_i^-)$. Note that $p(1-p)$ is an even number. \square

Proposition 3.3 implies Proposition 3.4.

Proposition 3.4. Let K be a $(2m+1)$ -dimensional closed oriented submanifold $\subset S^{2m+3}$. Let S be an $(m+1, m+1)$ -Seifert matrix. Then we have

$$S = (-1)^m \cdot {}^t S.$$

Let K be a $(4k+1)$ -dimensional spherical knot ($k \in \mathbb{N} \cup \{0\}$). We regard naturally $(H_{2k+1}(V; \mathbb{Z})/\text{Tor}) \otimes \mathbb{Z}_2$ as a subgroup of $H_{2k+1}(V; \mathbb{Z}_2)$. Then we can take a basis $\{x_1, \dots, x_{\nu}, y_1, \dots, y_{\nu}\}$ of $(H_{2k+1}(V; \mathbb{Z})/\text{Tor}) \otimes \mathbb{Z}_2$ such that $x_i \cdot x_j = 0$, $y_i \cdot y_j = 0$, $x_i \cdot y_j = \delta_{ij}$ for any pair (i, j) , where \cdot denotes the \mathbb{Z}_2 -intersection product. The Arf invariant of K is

$$\left(\sum_{i=1}^{\nu} \text{lk}(x_i, x_i^+) \cdot \text{lk}(y_i, y_i^+) \right) \bmod 2.$$

Let $L = (L_1, \dots, L_{\mu})$ be a $(4k+1)$ -link ($k \in \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{N} - \{1\}$). We define the Arf invariant of L . There are two cases.

- (1) Let $4k+1 \geq 5$. The Arf invariant of L is defined in the same manner as the knot case.
- (2) Let $4k+1 = 1$. See Appendix of [14] and [18, Note right above Note 1.2.1].

4. MAIN THEOREMS IN THE 2-DIMENSIONAL CASE

Two polynomials, $f(t)$ and $g(t)$, $\in \mathbb{Z}[t, t^{-1}]$ are said to be $\mathbb{Z}[t, t^{-1}]$ -balanced if there is an integer n such that $f(t) = \pm t^n \cdot g(t)$.

Theorem 4.1. *Let $L = (L_1, \dots, L_m)$ be a 2-dimensional closed oriented submanifold $\subset S^4$. Let each L_i be connected. Let g_i be the genus of L_i . Let $m = 1 + \sum_1^m g_i$. Let $S_\nu(V)$ be a positive ν -Seifert matrix associated with a Seifert hypersurface for L ($\nu = 1, 2$). Let $N_\nu(V)$ be its related negative ν -Seifert matrix. Let $t \cdot S_\nu(V) - N_\nu(V)$ be their related ν -Alexander matrix. Then the $\mathbb{Z}[t, t^{-1}]$ -balanced class of*

$$\det(t \cdot S_\nu(V) - N_\nu(V))$$

is a topological invariant of L .

Note. (1) We have $b_0(L) = \frac{1}{2}b_1(L) + 1$, where b_j is the j -th betti number.

(2) Since any Seifert hypersurface is connected by the definition, the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\det(t \cdot S_\nu(L) - N_\nu(L))$ is determined by the $\mathbb{Q}[t, t^{-1}]$ -module $H_1(\tilde{X}; \mathbb{Q})$, where \tilde{X} is the infinite cyclic covering space of L .

(3) Let $\Delta(t)$ be a polynomial which represents the ν - $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of L . By the definition, the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\Delta(t)$ is the ν - $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial associated with L .

Definition 4.2. The $\mathbb{Z}[t, t^{-1}]$ -balanced class defined in Theorem 4.1 is called the ν - $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of L .

We call a 2-dimensional closed oriented submanifold $L = (K_1, K_2) \subset S^4$ an (S^2, T^2) -link if K_1 (resp. K_2) is diffeomorphic to S^2 (resp. T^2). Note that $b_0(L) = \frac{1}{2}b_1(L) + 1$ holds.

Proposition 4.3. *There are (S^2, T^2) -links, $A = (A_1, A_2)$ and $B = (B_1, B_2) \subset S^4$ such that their ν - $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomials are not equivalent ($\nu = 1, 2$) and such that their ν - $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomials are equivalent.*

Recall the paragraph right before ‘Theorem 4.4 cited in §1’.

Theorem 4.4. *Let $L_+ = (L_{+,1}, \dots, L_{+,m_+})$ be a 2-dimensional closed oriented submanifold $\subset S^4$. Let each $L_{+,i}$ be connected. Let $g_{+,i}$ be the genus of $L_{+,i}$. Let $m_+ = 1 + \sum_1^{m_+} g_{+,i}$. Let (L_+, L_-, L_0) be a $(1, 2)$ -pass-move-triple. Then there is a polynomial $\Delta_{\nu, K_*}(t) (* = +, -, 0$ and $\nu = 1, 2)$ which represents the $\mathbb{Z}[t, t^{-1}]$ - ν -Alexander polynomial for K_* , and we have the identity*

$$\Delta_{\nu, L_+}(t) - \Delta_{\nu, L_-}(t) = (t - 1) \cdot \Delta_{\nu, L_0}(t).$$

Note. (1) If $m = 1$, K_+ and K_- are homeomorphic to S^2 . Then K_0 is homeomorphic to S^2 or $S^2 \amalg T^2$. In each case Theorem 4.4 holds. We do not need the condition on the homeomorphism type of K_0 .

(2) By [21, Proposition 4.3], we cannot normalize the $\mathbb{Z}[t, t^{-1}]$ - ν -Alexander polynomial for L_* to be compatible with the identities in Theorem 4.4. On the other hand, we can define the ‘normalized’ Alexander polynomial in a case of the $(4k + 1)$ -dimensional case so that it is compatible with a local-move-identity. See Definition 6.1 and Theorem 6.3 for detail.

(3) If we remove the condition on the betti number, the identity does not hold in general by [21, Proposition 4.2].

Proposition 4.5. *There are $(1, 2)$ -pass-move-triples $L = (L_+, L_-, L_0)$ and $L' = (L'_+, L'_-, L'_0)$ with the following properties:*

- (1) $b_0(L) = \frac{1}{2}b_1(L) + 1$. $b_0(L') = \frac{1}{2}b_1(L') + 1$.
- (2) $t - 1$ (resp. $t - 1, 1$) represents the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial of L_+ (resp. L_-, L_0).
- (3) $t - 1$ (resp. $t - 1, 1$) represents the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial of L'_+ (resp. L'_-, L'_0).
- (4) $4(t - 1)$ (resp. $3(t - 1), 1$) represents the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of L_+ (resp. L_-, L_0).
- (5) $2(t - 1)$ (resp. $t - 1, 1$) represents the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of L'_+ (resp. L'_-, L'_0).

Note. Take two arbitrary different polynomials from $4(t - 1)$, $3(t - 1)$, $2(t - 1)$, and $t - 1$. Then they are not $\mathbb{Z}[t, t^{-1}]$ -balanced but $\mathbb{Q}[t, t^{-1}]$ -balanced.

Proposition 4.6. *Let V be a Seifert hypersurface for an (S^2, T^2) -link $L = (K_1, K_2)$. Then we have the following:*

- (1) *There is a basis, $\{\tau_1, \dots, \tau_n\}$, of $H_2(V; \mathbb{Z})$, where n is a nonnegative integer.*
- (2) *There is a set $\{\sigma_1, \dots, \sigma_n\} \subset H_1(V; \mathbb{Z})$ such that $\{\pi(\sigma_1), \dots, \pi(\sigma_n)\}$ is a basis of $H_1(V; \mathbb{Z})/\text{Tor}$, where π is the natural projection homomorphism $H_1(V; \mathbb{Z}) \rightarrow H_1(V; \mathbb{Z})/\text{Tor}$.*
- (3) *The intersection product of σ_i and τ_j in V is
$$\begin{cases} 0 & \text{if } i = 1. \\ \delta_{ij} & \text{if } i \geq 2 \end{cases}$$*

Definition 4.7. Let $L = \{K_1, K_2\}$ be an (S^2, T^2) -link and V a Seifert hypersurface for L . Take sets, $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_n\}$, as in Proposition 4.6. We define the *pseudo-alinking number* of L to be the absolute value of the Seifert pairing of σ_1 and τ_1 .

Theorem 4.8. Let $L = (L_1, L_2)$ be an (S^2, T^2) -link $\subset S^4$. Let $\Delta_{1,L}(t)$ be a polynomial which represents the $\mathbb{Z}[t, t^{-1}]$ -1-Alexander polynomial for L . Then

$$\left| \frac{\Delta_{1,L}(t)}{(t-1)} \right|_{t=1}$$

is the pseudo-alinking number of L , where $|\cdot|$ denotes the absolute value.

Definition 4.9. ([24].) Let $L = (K_1, K_2)$ be an ordered closed oriented 2-dimensional submanifold $\subset S^4$. Let K_1 and K_2 be connected. Take any circle embedded in K_i . Give any orientation to the circle. Consider the linking number of the circle and K_j ($i \neq j$). Make a set of all of the linking number. Then the set is regarded as $n \cdot \mathbb{Z}$ for a number $n \in \{0\} \cup \mathbb{N}$. Note that if $n = 0$, then the set is $\{0\}$. We call this number n the *alinking number* $\text{alk}(K_i \subset L, K_j \subset L)$ of K_i in L around K_j in L . Note that $\text{alk}(K_1 \subset L, K_2 \subset L)$ is not equal to $\text{alk}(K_2 \subset L, K_1 \subset L)$ in general.

Note that if K_1 is diffeomorphic to S^2 , $\text{alk}(K_1 \subset L, K_2 \subset L)$ is zero. So, in this case, let the alinking number of L mean $\text{alk}(K_2 \subset L, K_1 \subset L)$.

Problem 4.10. Let $L = (K_1, K_2)$ be an (S^2, T^2) -link. Is the alinking number of L different from the pseudo-alinking number of L in general?

Note. (1) Note 7.9 is related to Problem 4.10.

(2) Proposition 7.10 claims that the alinking number is a ‘surface-link cobordism’ invariant. How about the pseudo-alinking number?

Theorem 4.11. Let $L = (K_1, K_2)$ be an (S^2, T^2) -link. Then the following three conditions are equivalent.

- (1) The alinking number of L is zero.
- (2) The pseudo-alinking number of L is zero.
- (3) $\left| \frac{\Delta_{1,L}(t)}{(t-1)} \right|_{t=1}$ is zero.

Theorem 4.12. Let $L = (K_1, K_2)$ be an (S^2, T^2) -link. Suppose that there is a Seifert hypersurface V such that $\text{Tor}H_1(V; \mathbb{Z}) \cong 0$. Then the alinking number of L is $\left| \frac{\Delta_{1,L}(t)}{(t-1)} \right|_{t=1}$.

Let $L = (K_1, K_2)$ be an (S^2, T^2) -link. We say that L is *ribbon* if there is an immersion $f : B \amalg H \looparrowright S^4$ with the following properties, where B is a 3-ball and H is a genus one handle body: The self-intersection of f consists of double points and is a disjoint union of 2-discs. Note that f^{-1} (each disc) is a disjoint union of two 2-discs. One of the two disc is included in the interior of $B \amalg H$. The intersection of $\partial(B \amalg H)$ and the other disc is the boundary of the other disc.

Corollary 4.13. *Let $L = (K_1, K_2)$ be a ribbon (S^2, T^2) -link. Then the alinking number of L is $\left| \frac{\Delta_{1,L}(t)}{(t-1)} \Big|_{t=1} \right|$.*

If (K_+, K_-, K_0) is a triple of 1-links as in §1 and if K_+ and K_- are 1-knots, then K_0 is always a 2-component 1-link. However the 2-dimensional case we have the following theorem.

Theorem 4.14. *There are $(1, 2)$ -pass-move-triples, $L = (L_+, L_-, L_0)$ and $L' = (L'_+, L'_-, L'_0)$, with the following properties:*

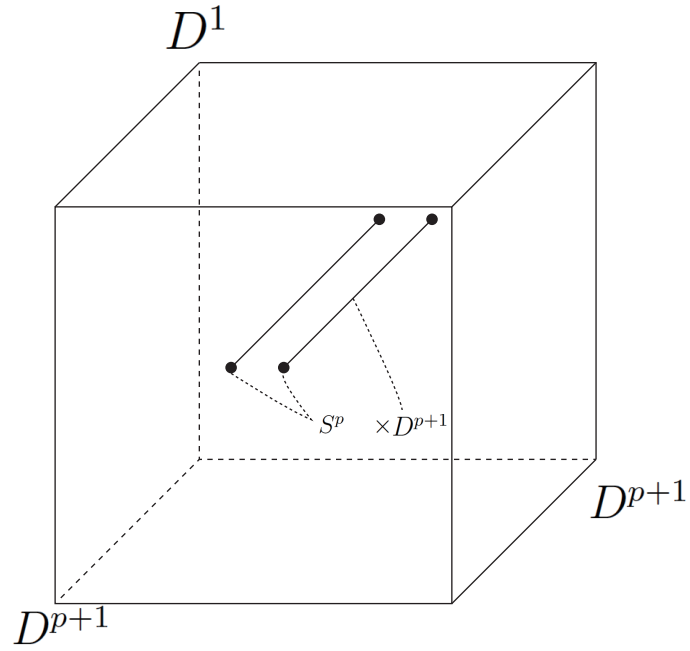
- (1) $b_0(L) = \frac{1}{2}b_1(L) + 1$. $b_0(L') = \frac{1}{2}b_1(L') + 1$.
- (2) L_+ and L'_+ are diffeomorphic to S^2 . Hence L_- and L'_- are diffeomorphic to S^2 .
- (3) The $\mathbb{Z}[t, t^{-1}]$ -1-Alexander polynomial of L_+ (resp. L_-) is equivalent to that of L'_+ (resp. L'_-).
- (4) L_0 is diffeomorphic to $S^2 \amalg T^2$. L'_0 is diffeomorphic to S^2 . Hence L_0 and L'_0 are not diffeomorphic.

See Corollary 6.7. In a $(4k + 1)$ -dimensional case we have similar situation to the 1-dimensional case, different from the 2-dimensional case.

5. REVIEW OF TWIST-MOVES ON HIGH DIMENSIONAL KNOTS

In the following section (§6) we have high dimensional analogues of §4. We prove a new local-move-identity for the ‘normalized’ Alexander polynomial of a kind of $(4k + 1)$ -dimensional closed oriented submanifolds $\subset S^{4k+3}$ (Theorem 6.3). The local-move-identity is associated with the twist-move, which is reviewed in this section. We introduce the ‘pseudo-twinkling number’ as an analogue of the pseudo-alinking number, the alinking number, and the linking number (Definition 6.5). We show a relation between the ‘normalized’ Alexander polynomial and the pseudo-twinkling number (Theorem 6.7). The pseudo-twinkling number is an analogue of the pseudo-alinking number but a relation between the pseudo-twinkling number and the ‘normalized’ Alexander polynomial in Corollary 6.8 is different from the relations between the pseudo-alinking number and the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial in §4.

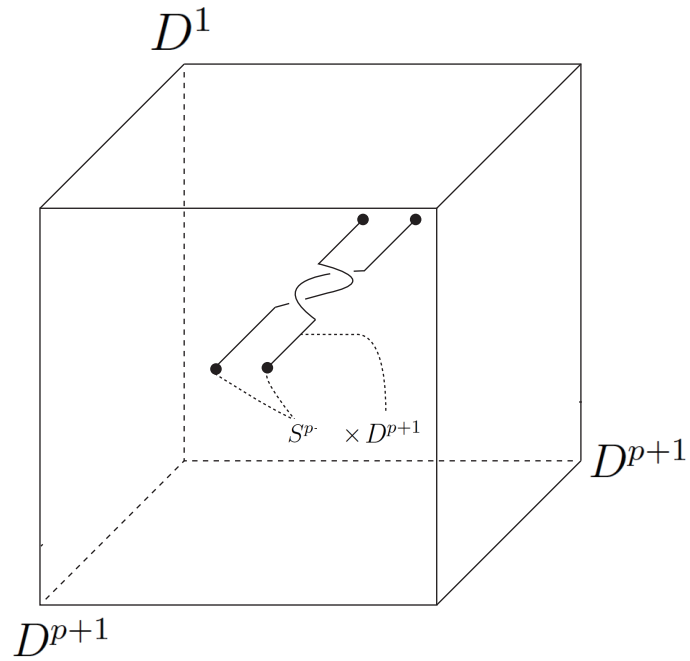
We review twist-moves on high dimensional knots in this section. (Note: In [21] the twist-move is called the *XXII*-move.) Figure 5.1, which consists of the three figures (1), (2) and (3), is a diagram of a twist-move-triple. Confirm the following: if $p = 0$, the twist-move is the crossing change on 1-links and Figure 5.1 is one drawn in the first paragraph in §1.



This cube is $D^{2p+3} = B$.

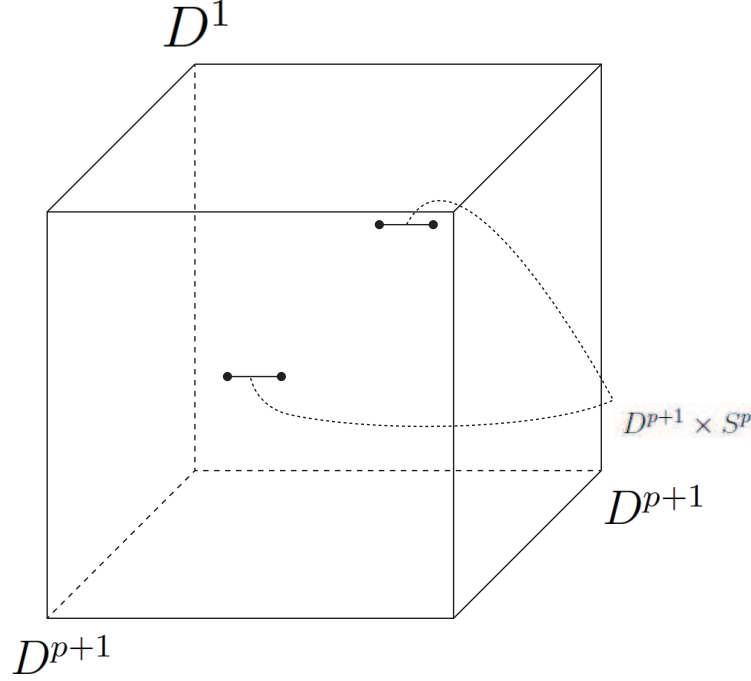
$B \cap K_+$

Figure 5.1.(1): A twist-move-triple



$B \cap K_-$

Figure 5.1.(2): A twist-move-triple



$$B \cap K_0$$

Figure 5.1.(3): **A twist-move-triple**

Let K_+ , K_- , K_0 be $(2p+1)$ -dimensional closed oriented submanifolds $\subset S^{2p+3}$ ($p \in \mathbb{N} \cup \{0\}$). Let B be a $(2p+3)$ -ball trivially embedded in S^{2p+3} . Suppose that K_+ coincides with K_- (resp. K_0) in $\overline{S^{2p+3} - B}$. Take a single $(2p+2)$ -dimensional $(p+1)$ -handle h_+ (resp. h_-) embedded in B such that $(\text{the handle}) \cap \partial B$ is the attaching part of the handle.

Note. [4, 5, 25, 26, 27] etc. imply that the core of h_+ (resp. h_-) is trivially embedded in B under the above condition.

Suppose that $(h_+ - \text{its attaching part}) \cap (h_- - \text{its attaching part}) = \emptyset$. Suppose that their attaching parts coincide. Thus we can suppose that we regard $h_+ \cup h_-$ as an oriented $(2p+2)$ -submanifold $\subset S^{2p+3}$ if we give the opposite orientation to h_- . Then we can define a $(p+1)$ -Seifert matrix for the $(2p+2)$ -submanifold $h_+ \cup h_-$. We can suppose that the $(p+1)$ -Seifert matrix of $\partial(h_+ \cup h_-)$ associated with $h_+ \cup h_-$ is a 1×1 -matrix (1).

Note 5.1. In the case of the twist-move on the $(4k+1)$ -dimensional submanifolds we can distinguish K_+ from K_- because the Seifert matrix is a 1×1 -matrix (1) even if we change the orientation of $h_+ \cup h_-$. On the other hand, in the $(1, 2)$ -pass-move-triple case we cannot distinguish K_+ from K_- . See Note 2.2.

Note. Suppose that p is an odd natural number, and let $p = 2k+1$. The twist-move for $(4k+3)$ -submanifolds $\subset S^{4k+5}$ ($4k+3 \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$) has the following property: Suppose that K_+ is made into K_- by the twist-move. Suppose that K_+ is PL homeomorphic to the standard sphere. Then $H_*(K_-; \mathbb{Z})$ is not congruent to $H_*(K_+; \mathbb{Z})$

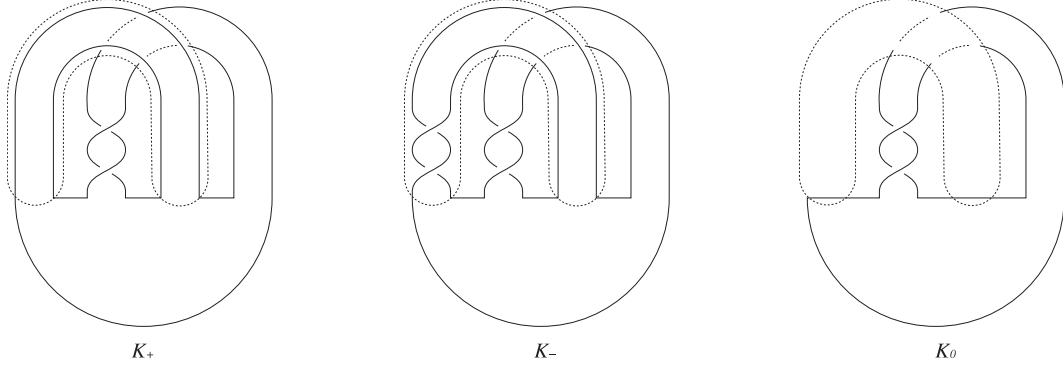


Figure 5.2: A twist-move-triple of $(4k + 1)$ -knots.

in general. Example: Make a Seifert hypersurface V_* for a 3-knot K_* ($* = +, -$) as follows. A framed link representation of V_+ is the Hopf link such that the framing of one component is zero and such that that of the other is two. A framed link representation of V_- is the Hopf link such that the framing of each component is two.

Let K_* ($* = +, -$) satisfy that $K_* \cap \text{Int}B = (\partial h_* - \partial B)$. Note the following. When we define K_+ , h_+ exists in B and h_- does not exist in B . When we define K_- , h_- exists in B and h_+ does not exist in B . Let $P = K_+ \cap (S^{2p+3} - \text{Int}B)$. Let $Q = h_+ \cap \partial B$. Let $T = P \cup Q$. Then T is an $(2p + 1)$ -dimensional oriented closed submanifold in $S^{2p+3} - \text{Int}B$. Let K_0 be T in S^{2p+3} . Then we say that an ordered set (K_+, K_-, K_0) is related by a single *twist-move*. (K_+, K_-, K_0) is called a *twist-move-triple*. We say that K_- (resp. K_+) is obtained from K_+ (resp. K_-) by a single *negative-twist-move* (resp. *positive-twist-move*) in B .

See Figure 5.2 for a twist-move-triple of $(4k + 1)$ -knots.

Note. In the twist-move in the $(4k + 1)$ -dimensional case the homotopy type of K_0 is determined if K_+ is homotopy type equivalent to S^{4k+1} by [1, 7]. On the other hand, in the $(1, 2)$ -pass-move-triple case the homotopy type of K_0 is not determined even if K_+ is diffeomorphic to S^2 . See Note to Theorem 4.4.

Let (K_+, K_-, K_0) be related by a single twist-move in B . Then there is a Seifert hypersurface V_* for K_* ($* = +, -, 0$) with the following properties.

- (1) $V_{\#} = V_0 \cup h_{\#}$ ($\# = +, -$). $V_{\#} \cap B = h_{\#}$.
 - (2) $V_0 \cap \text{Int} B = \phi$. $V_0 \cap \partial B$ is the attaching part of $h_{\#}^p$.
- (The idea of the proof is the Thom-Pontrjagin construction.)

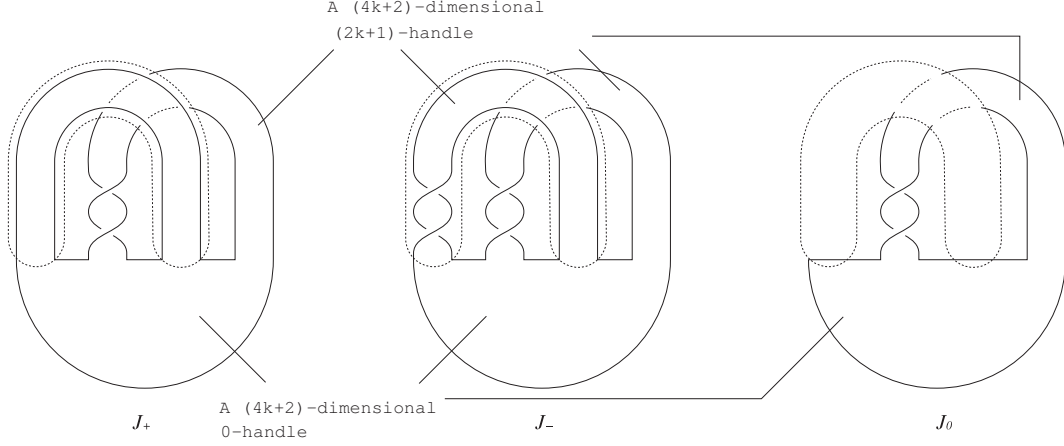


Figure 5.3: A twist-move-triple of Seifert hypersurfaces for $(4k + 1)$ -knots

The ordered set (V_+, V_-, V_0) is called a *twist-move-triple of Seifert hypersurfaces* for (K_+, K_-, K_0) . We say that V_- (resp. V_+) is obtained from V_+ (resp. V_-) by a single *negative-twist-move* (resp. *positive-twist-move*) in B .

See Figure 5.3 for a twist-move-triple of Seifert hypersurfaces for $(4k + 1)$ -knots.

In [17, 21] we introduced the (p, q) -pass-move, which is a kind of local-moves. we found local-move-identities of the Alexander polynomial associated with the (p, q) -pass-move. We showed other relations between some invariants of knots and the (p, q) -pass-move. In [11] we also proved such new results.

6. MAIN THEOREMS IN THE $4k+1$ DIMENSIONAL CASE

We can define the ‘normalized’ Alexander polynomial in a case of the $(4k+1)$ -dimensional case so that it is compatible with a local-move-identity associated with the twist-move (see Definition 6.1 and Theorem 6.3 for detail). On the other hand, in the 2-dimensional case we cannot normalize the $\mathbb{Z}[t, t^{-1}]$ -Alexander module so that it is compatible with the $(1, 2)$ -pass-move-identity (see [21, Proposition 4.3]).

Definition 6.1. Let $k \in \{0\} \cup \mathbb{N}$. Let K be a $(4k + 1)$ -dimensional closed oriented submanifold $\subset S^{4k+3}$ whose homotopy type is S^{4k+1} . Let V be a Seifert hypersurface for K . Let $S_{2k+1}(V)$ be a $(2k+1)$ -Seifert matrix and $N_{2k+1}(V)$ its related $(2k+1)$ -negative Seifert matrix associated with a Seifert hypersurface V for K . Call

$$\hat{\Delta}_K(t) = \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V))$$

the *normalized Alexander polynomial* for K .

Let $k \in \{0\} \cup \mathbb{N}$. Let K be a $(4k+1)$ -dimensional closed oriented submanifold $\subset S^{4k+3}$ whose homotopy type is $S^{2k+1} \times S^{2k}$. Let $S_{2k+1}(V)$ and $N_{2k+1}(V)$ be defined in the same manner as in the previous paragraph. Define the *normalized Alexander polynomial* $\hat{\Delta}_K(t)$ for K to be

$$\begin{cases} \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V)) & \text{if a } 2k\text{-Alexander matrix associated with } V \text{ induces an injective map on} \\ & H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q}), \\ 0 & \text{else.} \end{cases}$$

Note. (1) Recall that any $2k$ -Alexander matrix associated with V induces a homomorphism $H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q})$ as in Proof of Proposition 3.2.

(2) By the definition of Alexander matrices we have the following: If a $2k$ -Alexander matrix associated with V induces (resp. does not induce) an injective map, then any (resp. no) $2k$ -Alexander matrix associated with V induces an injective map.

Theorem 6.2. *The normalized Alexander polynomial $\hat{\Delta}_K(t)$ does not depend on the choice of V , and hence is a topological invariant.*

Theorem 6.3. *Let K_+ be a $(4k+1)$ -knot $\subset S^{4k+3}$. Let (K_+, K_-, K_0) be a twist-move-triple. Then*

$$\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t) = (t^{\frac{1}{2}} - t^{\frac{-1}{2}}) \cdot \hat{\Delta}_{K_0}(t),$$

where $\hat{\Delta}_K(t)$ denotes the normalized Alexander polynomial of K .

See Figure 5.2 for an example of a twist-move-triple of $(4k+1)$ -knots which satisfy the identity in Theorem 6.3. There, we regard $S(V_*)$, $N(V_*)$, and $\hat{\Delta}_{K_*}(t)$ as follows.

$$\begin{aligned} S(V_+) &= \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, N(V_+) = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, \hat{\Delta}_{K_+}(t) = 1, \\ S(V_-) &= \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, N(V_-) = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \hat{\Delta}_{K_-}(t) = t + \frac{1}{t} - 1, \\ S(V_0) &= (-1), N(V_0) = (-1), \text{ and } \hat{\Delta}_{K_0}(t) = -t^{\frac{1}{2}} + t^{\frac{-1}{2}}. \end{aligned}$$

We say that $x \in H_i(X; \mathbb{Z})$ is *order finite* (resp. *order infinite*) if $x \in \text{Tor}H_i(X; \mathbb{Z})$ (resp. $\notin \text{Tor}H_i(X; \mathbb{Z})$). Suppose that $x \in H_i(X; \mathbb{Z})$ is nonzero and order finite. Let p be the minimum number of $\{n \in \mathbb{N} | nx = 0\}$. Then we say that x is *order p* . We say that x is *order zero* if $x = 0 \in H_i(X; \mathbb{Z})$.

Definition 6.4. We say that $x \in H_i(X; \mathbb{Z})$ is *divisible* if x is order infinite and if there is $y \in H_i(X; \mathbb{Z})$ such that $x = ny$ for an integer n with the condition $|n| > 1$. We suppose that x is order infinite when we say that $x \in H_i(X; \mathbb{Z})$ is divisible (resp. non-divisible). If $y \in H_i(X; \mathbb{Z})$ is order infinite, there is a non-divisible i -cycle $z \in H_i(X; \mathbb{Z})$ such that

there is an integer m with the condition $y = mz$ (m may be ± 1). Call z a *non-divisible i -cycle associated with y* .

Definition 6.5. Let $k \in \{0\} \cup \mathbb{N}$. Let K be a $(4k+1)$ -dimensional closed oriented submanifold $\subset S^{4k+3}$ whose homotopy type is $S^{2k} \times S^{2k+1}$. Let V be a Seifert hypersurface for K . We define the *pseudo-twinkling number* of K to be

$$\begin{cases} s(\tau, \tau) & \text{if there is a non-divisible } (2k+1)\text{-cycle } \tau \subset V \text{ such that} \\ & \text{for any } (2k+1)\text{-cycle } \alpha \subset V \text{ the intersection product } \tau \cdot \alpha \text{ in } V \text{ is zero,} \\ 0 & \text{else,} \end{cases}$$

where $s(\alpha, \beta)$ denotes the Seifert paring of $(2k+1)$ -cycles α and β . Note that if $k = 0$, the twinkling number is the linking number.

Note. We would define the ‘twinkling number’ to be $s(\gamma, \gamma)$, where γ is a generator of $H_{2k+1}(S^{2k} \times S^{2k+1})$. So we call the above one the pseudo-twinkling number by an analogy of the relation between the alinking number and the pseudo-alinking number although we do not discuss the twinkling number so much in this paper. The author does not know whether the twinkling number and the pseudo-twinkling number are non-equivalent in general. He could prove that if there is a Seifert hypersurface V such that $\text{Tor}H_*(V; \mathbb{Z}) \cong 0$, they are equivalent. Note 8.13 is related to this question. He thinks that we have results which are analogues of Theorem 4.12 and Corollary 4.13. He could prove that the pseudo-twinkling number is ‘submanifold-cobordism’ invariant, where submanifold-cobordism is defined in a similar fashion to that of knot cobordism by using $(\text{the submanifold}) \times [0, 1]$. (See the definition right before Proposition 7.10 for an example of submanifold-cobordism.) He does not think that the twinkling number is ‘submanifold-cobordism’ invariant.

Proposition 6.6. *The pseudo-twinkling number of K does not depend on the choice of V and that of τ , and hence is a topological invariant.*

Theorem 6.7. *Let K be a $(4k+1)$ -dimensional closed oriented submanifold $\subset S^{4k+3}$ whose homotopy type is $S^{2k} \times S^{2k+1}$. Let $\hat{\Delta}_K(t)$ be the normalized Alexander polynomial of K . Then the pseudo-twinkling number of K is*

$$\left. \frac{\hat{\Delta}_K(t)}{t^{\frac{1}{2}} - t^{\frac{-1}{2}}} \right|_{t=1}$$

Proposition 6.6 and Theorem 6.7 imply the following.

Corollary 6.8. *Let K_+ be a $(4k+1)$ -knot $\subset S^{4k+3}$. Let (K_+, K_-, K_0) be a twist-move-triple. Then the pseudo-twinkling number of K_0 is*

$$\left. \frac{\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t)}{(t^{\frac{1}{2}} - t^{\frac{-1}{2}})^2} \right|_{t=1}$$

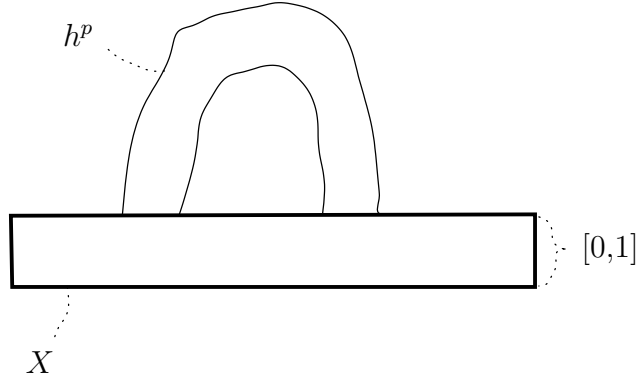


Figure 7.1: A handle h^p is attached to $X \times [0, 1]$.

Note. (1) Compare ‘Theorem 6.7 and Corollary 6.8’ with ‘Theorem 4.8 and Theorem 4.14’.

(2) There is a relation among $\hat{\Delta}_K(t)$, the bP-subgroup, and the inertia group by way of [21, Theorem 3.4] and Corollary 6.8.

Other results in [11, 21] written in the $\mathbb{Q}[t, t^{-1}]$ -term could be generalized into $\mathbb{Z}[t, t^{-1}]$ -term in some fashion without difficulty although we must take care of [11, §10].

7. PROOF OF RESULTS IN §4

Definition 7.1. Let X be an x -dimensional submanifold of an m -dimensional manifold M ($x, m \in \mathbb{N}, x < m$). Suppose that we can embed $X \times [0, 1]$ in M so that $X \times \{0\} = X$. Suppose that an $(x + 1)$ -dimensional handle h^p is embedded in M and is attached to $X \times [0, 1]$ ($p \in \mathbb{N} \cup \{0\}, 0 \leq p \leq x$). Suppose that the attaching part of h^p is embedded in $X \times \{1\}$. See Figure 7.1. Suppose that $h^p \cap (X \times [0, 1])$ is only the attaching part of h^p . Let $X' = \overline{\partial(h^p \cup (X \times [0, 1]))} - (X \times \{0\})$. Note that there are two cases, $\partial X = \emptyset$ and $\partial X \neq \emptyset$. Then we say that X' is obtained from X by *the surgery by using the embedded handle h^p* . We do not say that we use $X \times [0, 1]$ if there is no danger of confusion.

Note. Of course we can define ‘embedded surgery’ even if we cannot embed $X \times [0, 1]$ in M . However we do not need the case in this paper.

Proof of Theorem 4.1. Let V and V' be Seifert hypersurfaces for L . Recall that V and V' are connected by the definition. It suffices to prove that the ν - $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial ($\nu = 1, 2$) defined by using V is the same as that defined by using V' .

By the same manner as that in [16, sections 4 and 5], and that in [13, Proof of Claim 8.1], we have the following: There are (not necessarily connected) 3-dimensional compact oriented submanifolds $V = U_1, U_2, \dots, U_{u-1}, U_u = V' \subset S^4$ ($u \in \mathbb{N}$) such that $\partial U_* = L$ and

such that U_{*+1} is obtained from U_* ($2 \leq *+1 \leq u$) by a surgery by using an embedded 4-dimensional handle.

If some of U_{\natural} are not connected, use 4-dimensional 1-handles and then we can suppose that all U_{\natural} are connected, that is, all U_{\natural} are Seifert hypersurfaces for L .

Therefore it suffices to prove the following case: V' is obtained from V by a surgery by using an embedded 4-dimensional i -handle h^i ($i = 1, 2, 3$).

Lemmas 7.2 and 7.3 imply Theorem 4.1.

Lemma 7.2. *Theorem 4.1 holds in the case $i = 1, 3$.*

Proof of Lemma 7.2. $V' = V \sharp (S^1 \times S^2)$ or $V = V' \sharp (S^1 \times S^2)$ where \sharp is the connected-sum. If $V' = V \sharp (S^1 \times S^2)$, an Alexander matrix $A(t)$ for V is related to an Alexander matrix $A'(t)$ for V' as follows.

$$A(t) = \left(\begin{array}{c|cccc} t \text{ or } -t & * & \cdot & \cdot & * \\ \hline 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{array} \middle| \begin{array}{c} A'(t) \end{array} \right),$$

where $A(t)$ is an $n \times n$ -matrix and $A'(t)$ is an $(n-1) \times (n-1)$ -matrix ($n \in \mathbb{N}$). If $V = V' \sharp (S^1 \times S^2)$, an Alexander matrix $A(t)$ for V is related to an Alexander matrix $A'(t)$ for V' as follows.

$$A'(t) = \left(\begin{array}{c|cccc} t \text{ or } -t & * & \cdot & \cdot & * \\ \hline 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{array} \middle| \begin{array}{c} A(t) \end{array} \right),$$

where $A'(t)$ is an $n \times n$ -matrix and $A(t)$ is an $(n-1) \times (n-1)$ -matrix.

Hence $\det A(t)$ is $\mathbb{Z}[t, t^{-1}]$ -balanced to $\det A'(t)$ in the both cases.

This completes the proof of Lemma 7.2. □

Lemma 7.3. *Theorem 4.1 holds in the case $i = 2$.*

Proof of Lemma 7.3. Let $C \subset V$ be the core of the attaching part of h^2 . Let $N(C)$ be the tubular neighborhood of C in V . By Definition 7.1, there is $W = (V \times [0, 1]) \cup h^2$ which is embedded in S^4 . Let $C' \subset V'$ be the core of the attaching part of the dual handle \bar{h}^2 of h^2 . Note that \bar{h}^2 is a 4-dimensional 2-handle. Recall that \bar{h}^2 is attached to V' and that $W = (V \times [0, 1]) \cup h^2 = (V' \times [0, 1]) \cup \bar{h}^2$.

There are two cases:

- (1) $[C] \in H_1(V; \mathbb{Z})$ is order finite.

(2) $[C] \in H_1(V; \mathbb{Z})$ is order infinite.

We divide these two cases into four cases.

(1-1) $[C] \in H_1(V; \mathbb{Z})$ is order finite. $[C'] \in H_1(V; \mathbb{Z})$ is order finite.

(1-2) $[C] \in H_1(V; \mathbb{Z})$ is order finite. $[C'] \in H_1(V; \mathbb{Z})$ is order infinite.

(2-1) $[C] \in H_1(V; \mathbb{Z})$ is order infinite. For any closed oriented surface F embedded in V , it holds that the intersection product of $[F] \in H_2(V, \partial V; \mathbb{Z})$ and $[C] \in H_1(V; \mathbb{Z})$ in V is zero.

(2-2) $[C] \in H_1(V; \mathbb{Z})$ is order infinite. There is a closed oriented surface F embedded in V such that the intersection product of $[F] \in H_2(V, \partial V; \mathbb{Z})$ and $[C] \in H_1(V; \mathbb{Z})$ in V is nonzero.

Lemmas 7.4, 7.6, 7.7, and 7.8 imply Lemma 7.3.

Lemma 7.4. *Lemma 7.3 holds in the (1-1) case.*

Proof of Lemma 7.4. Proposition 7.5 implies that the (1-1) case does not occur.

Proposition 7.5. *If $[C] \in H_1(V; \mathbb{Z})$ is order finite, then $C' \in H_1(V'; \mathbb{Z})$ is order infinite.*

Proof of Proposition 7.5. Let $[C]$ (resp. $[C']$) be order q (resp. q'), where $q, q' \in \mathbb{N} \cup \{0\}$. Take a 2-chain $\alpha \subset V$ (resp. $\alpha' \subset V'$) whose boundary is $q \cdot C$ (resp. $q' \cdot C'$).

Hence $\alpha \cup ((q \cdot C) \times [0, 1]) \cup (q \cdot (\text{the core of } h^2))$ (resp. $\alpha' \cup ((q' \cdot C') \times [0, 1]) \cup (q' \cdot (\text{the core of } \bar{h}^2))$) is a 2-cycle β (resp. β') $\subset W$. Note that $(q \cdot C) \times [0, 1] \hookrightarrow V \times [0, 1]$ (resp. $(q' \cdot C') \times [0, 1] \hookrightarrow V' \times [0, 1]$) is a level preserving embedding map. Recall that $(V \times [0, 1]) \cup h^2$ is diffeomorphic to $(V' \times [0, 1]) \cup \bar{h}^2$. Therefore the intersection product of β and β' in W is nonzero. However, since W is embedded in S^4 , this intersection product is zero. We arrived at a contradiction.

This completes the proof of Proposition 7.5. □

This completes the proof of Lemma 7.4. □

Lemma 7.6. *If Lemmas 7.7 and 7.8 hold, Lemma 7.3 holds in the (1-2) case.*

Proof of Lemma 7.6. Replace V with V' , h^2 with \bar{h}^2 , and C with C' . Therefore the (1-2) case is true if the (2) case is true. Note that the (2) case consists of the (2-1) case and the (2-2) case.

This completes the proof of Lemma 7.6. □

Lemma 7.7. *Lemma 7.3 holds in the (2-1) case.*

Proof of Lemma 7.7. We can suppose the following: There are a positive p -Seifert matrix $S_p(V)$ and its related negative p -Seifert matrix $N_p(V)$ ($p = 1, 2$) associated with V such that a square matrix $t \cdot S_p(V) - N_p(V)$ has a row all of whose elements are zero as follows:

$$\begin{pmatrix} 0 & \cdots & 0 \\ * & \cdots & * \\ \cdot & \cdots & \cdot \\ * & \cdots & * \end{pmatrix}.$$

Hence $\det(t \cdot S_p(V) - N_p(V)) = 0$. Hence the $\mathbb{Q}[t, t^{-1}]$ - p -Alexander polynomial is the $\mathbb{Q}[t, t^{-1}]$ -balanced class of zero. Hence we have the following: Let $S_p(V')$ be a positive p -Seifert matrix and $N_p(V')$ its related negative p -Seifert matrix ($p = 1, 2$) associated with V' . By Proposition 3.2 and Notes (2) and (3) to Theorem 4.1, it holds that $\det(t \cdot S_p(V') - N_p(V'))$ is $\mathbb{Q}[t, t^{-1}]$ -balanced to zero. Hence $\det(t \cdot S_p(V') - N_p(V')) = 0$.

Hence $\det(t \cdot S_p(V) - N_p(V))$ and $\det(t \cdot S_p(V') - N_p(V'))$ are not only $\mathbb{Q}[t, t^{-1}]$ -balanced but also $\mathbb{Z}[t, t^{-1}]$ -balanced.

This completes the proof of Lemma 7.7. \square

Lemma 7.8. *Lemma 7.3 holds in the (2-2) case.*

Proof of Lemma 7.8. Note that $W = (V \times [0, 1]) \cup h^2$ is diffeomorphic to $(V' \times [0, 1]) \cup \bar{h}^2$. Consider the exact sequence by a pair $((V \times [0, 1]) \cup h^2, V)$, where we regard V as $V \times \{0\}$:

$$\cdots \rightarrow H_*(V; \mathbb{Z}) \rightarrow H_*((V \times [0, 1]) \cup h^2; \mathbb{Z}) \rightarrow H_*((V \times [0, 1]) \cup h^2, V; \mathbb{Z}) \rightarrow \cdots$$

and the exact sequence by a pair $((V' \times [0, 1]) \cup \bar{h}^2, V')$, where we regard V' as $V' \times \{0\}$:

$$\cdots \rightarrow H_*(V'; \mathbb{Z}) \rightarrow H_*((V' \times [0, 1]) \cup \bar{h}^2; \mathbb{Z}) \rightarrow H_*((V' \times [0, 1]) \cup \bar{h}^2, V'; \mathbb{Z}) \rightarrow \cdots$$

By the existence of F , $[C'] \in H_{2k}(V'; \mathbb{Z})$ is order finite.

Let $\xi_1 \in H_1(V; \mathbb{Z})$ be a non-divisible 1-cycle associated with $[C]$. Let $\eta_1 \in H_2(V; \mathbb{Z})$ be a non-divisible 2-cycle associated with $[F]$. We can suppose the following:

- (1) There is a set $\{\xi_1, \xi_2, \dots, \xi_n\} \subset H_1(V; \mathbb{Z})$, where $n \in \mathbb{N} \cup \{0\}$. A set $\{\pi(\xi_1), \pi(\xi_2), \dots, \pi(\xi_n)\}$ is a basis of $H_1(V; \mathbb{Z})/\text{Tor}$, where π is the natural epimorphism $H_1(V; \mathbb{Z}) \rightarrow H_1(V; \mathbb{Z})/\text{Tor}$.
- (2) We can regard $\{\xi_2, \dots, \xi_n\} \subset H_1(V'; \mathbb{Z})$. $\{\pi(\xi_2), \dots, \pi(\xi_n)\}$ is a basis of $H_1(V'; \mathbb{Z})/\text{Tor}$.
- (3) There is a basis $\{\eta_1, \eta_2, \dots, \eta_n\}$ of $H_2(V; \mathbb{Z})$.
- (4) We can regard $\{\eta_2, \dots, \eta_n\}$ as a basis of $H_2(V; \mathbb{Z})$.
- (5) Since $H_*(\partial V; \mathbb{Z})$ is torsion free, the intersection product $\xi_i \cdot \eta_j$ in V (resp. in V') is

$$\begin{cases} 1 & \text{if } i = j = 1 \\ \delta_{ij} \text{ or zero} & \text{else.} \end{cases}$$

Hence we have the following: An Alexander matrix $A(t)$ for V is associated with $\{\xi_1, \dots, \xi_n\}$ and $\{\eta_1, \dots, \eta_n\}$. An Alexander matrix $A'(t)$ for V' is associated with $\{\xi_2, \dots, \xi_n\}$ and $\{\eta_2, \dots, \eta_n\}$. $A(t)$ is an $n \times n$ -matrix. $A'(t)$ is an $(n-1) \times (n-1)$ -matrix. Seifert pairings $s(\xi_*, \eta_{\#})$ ($2 \leq *$ and $2 \leq \#$) are not changed when we attach the 4-dimensional 2-handle h^2 to V .

$$A(t) = \left(\begin{array}{c|cccc} t \text{ or } -t & 0 & \cdot & \cdot & 0 \\ \hline * & & & & \\ \cdot & & & & \\ \cdot & & & & \\ * & & & & \end{array} \middle| \begin{array}{c} A'(t) \end{array} \right),$$

Hence $\det A(t)$ is $\mathbb{Z}[t, t^{-1}]$ -balanced to $\det A'(t)$.

This completes the proof of Lemma 7.8. □

This completes the proof of Lemma 7.3. □

This completes the proof of Theorem 4.1. □

Note. It is important that we can suppose that $\xi_1 \cdot \eta_1 = 1$. If it does not hold, $\det A(t)$ is not $\mathbb{Z}[t, t^{-1}]$ -balanced to $\det A'(t)$ in general. See the example in [11, §10].

Proof of Theorem 4.4. In [21, Proof of Theorem 4.1] we proved that there is a ν -Alexander matrix $A_{\nu, L_*}(t)$ for L_* ($*$ = +, −, 0 and $\nu = 1, 2$) such that

$$\det A_{\nu, L_+}(t) - \det A_{\nu, L_-}(t) = (t - 1) \cdot \det A_{\nu, L_0}(t).$$

This fact and Theorem 4.1 imply Theorem 4.4. □

Proof of Proposition 4.6. Consider the following exact sequence by a pair $(V, \partial V)$ (Note that $\partial V = S^2 \amalg T^2$. Here, S^2 denotes K_1 and T^2 K_2 .):

$$\cdots \xrightarrow{\partial} H_*(S^2 \amalg T^2; \mathbb{Z}) \xrightarrow{\iota} H_*(V; \mathbb{Z}) \xrightarrow{\rho} H_*(V, \partial V; \mathbb{Z}) \xrightarrow{\partial} H_{*-1}(S^2 \amalg T^2; \mathbb{Z}) \xrightarrow{\iota} \cdots.$$

We can take sets, $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_n\}$, to satisfy the conditions (1)-(3) in Proposition 4.6. □

Proof of Theorem 4.8. Take sets, $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_n\}$, as in Proposition 4.6. Then the 1-Alexander matrix $A(t)$ associated with the ordered sets, $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_n\}$, is written as follows:

$$\left(\begin{array}{c|cccc} (t-1) \cdot a_{11} & (t-1) \cdot a_{12} & \cdot & \cdot & (t-1) \cdot a_{1n} \\ \hline (t-1) \cdot a_{21} & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (t-1) \cdot a_{n1} & & & & \end{array} \middle| \begin{array}{c} X(t) \end{array} \right),$$

where we have the following: $a_{ij} = s(\sigma_i, \tau_j)$. $X(1) = \delta_{ij}$. $|a_{11}|$ is the pseudo-alinking number. Hence

$$A(t) = (t-1) \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ \hline (t-1) \cdot a_{21} & & & & \\ \cdot & & & & \\ \cdot & & & & \\ (t-1) \cdot a_{n1} & & & & \end{array} \middle| \begin{array}{c} X(t) \end{array} \right).$$

Hence

$$\frac{A(t)}{t-1} \Big|_{t=1} = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ \hline 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{array} \middle| \begin{array}{c} X(1) \end{array} \right).$$

Hence $\left| \frac{\det A(t)}{t-1} \right|_{t=1}$ is $|a_{11}|$, and hence is the pseudo-alinking number.

This completes the proof of Theorem 4.8. \square

Proof of Theorem 4.11. Theorem 4.8 implies that (2) \Leftrightarrow (3).

We prove that (1) \Leftrightarrow (2). By the exact sequence in Proof of Proposition 4.6, we have $H_2(V; \mathbb{Z}) \cong \mathbb{Z}^n$, $H_2(V, \partial V; \mathbb{Z}) \cong \mathbb{Z}^n$, $H_1(V; \mathbb{Z}) \cong \mathbb{Z}^n \oplus T$, $H_2(V, \partial V; \mathbb{Z}) \cong \mathbb{Z}^n \oplus T$, where T is the torsion part.

Note that $\rho : H_1(V; \mathbb{Z}) \xrightarrow{\rho} H_1(V, \partial V; \mathbb{Z})$ is not an isomorphism in general. See Note 7.9.

There is a nonzero element $\alpha \in H_1(S^2 \amalg T^2; \mathbb{Z})$ such that $\iota(\alpha)$ is order finite, where ι is the homomorphism in the exact sequence in Proof of Proposition 4.6. Note that α is represented by an embedded circle $\subset T^2$, and let the circle also be called α . Let β be an embedded circle in T^2 such that α intersects β transversely at one point. The 1-cycle which represented by β is also called β .

We prove $\iota(\beta)$ is order infinite in V . Reason: Suppose that $\iota(\beta)$ is order finite. Let P (resp. Q) be a 2-cycle $\subset V$ whose boundary is α (resp. β). We can suppose that P intersects Q transversely. Take $\partial(P \cap Q)$. It is a boundary of a 1-cycle $P \cap Q$ and hence it is zero $\in H_0(T^2)$. However it is one point by the definition of β hence it is not zero $\in H_0(T^2)$. We arrived at a contradiction.

Take $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_n\}$ as in Proposition 4.6. Since $\beta \in H_1(S^2 \amalg T^2; \mathbb{Z})$, $\iota(\beta) \cdot \tau_* = 0$ for all $*$. Hence σ_1 is a non-divisible 1-cycle associated with $\iota(\beta)$. Hence $\iota(\beta) = k \cdot \sigma_1$ for a nonzero integer k .

Since $S^2 \subset \partial V$ (recall that S^2 denotes K_1), the intersection product $\iota(S^2) \cdot \sigma_* = 0$ for any $*$. Since $H_2(V, \partial V; \mathbb{Z})$ is torsion-free, $\iota(S^2) = \tau_1$.

Hence $|\text{lk}(\beta, \tau_1)| = |\text{lk}(\iota(\beta), \tau_1)| = |k \cdot \text{lk}(\sigma_1, \tau_1)|$.

Suppose that the alinking number of L is zero. Hence $\text{lk}(\beta, \tau_1) = 0$. Hence the pseudo-alinking number $|\text{lk}(\sigma_1, \tau_1)|$ is zero.

Suppose that the pseudo-alinking number $|\text{lk}(\sigma_1, \tau_1)|$ is zero. Hence $\text{lk}(\beta, \tau_1) = 0$. We can use $\{\alpha, \beta\}$ as a basis of $H_1(T^2; \mathbb{Z})$. Since $\iota(\alpha)$ is order finite in V , $\text{lk}(\alpha, \tau_1) = 0$. Hence $\text{lk}(l \cdot \alpha + m \cdot \beta, \tau_1) = 0$ for any pair of integers (l, m) . Hence the alinking number of L is zero.

Hence (1) \Leftrightarrow (2).

This completes of the proof of Theorem 4.11. \square

Note 7.9. If we define V as in (1), $\iota : H_1(\partial V; \mathbb{Z}) \rightarrow H_1(V; \mathbb{Z})$ has the property in (2).

(1) Let $f : S^1 \hookrightarrow S^1 \times S^2$ be an embedding such that $f_* : H_1(S^1; \mathbb{Z}) \rightarrow H_1(S^1 \times S^2 - \text{Int}B^3; \mathbb{Z})$ carries 1 to p ($|p| > 1$, $p \in \mathbb{Z}$). Let B be an embedded 3-ball in $S^1 \times S^2$ such that $B \cap N(f(S^1)) = \emptyset$, where $N(f(S^1))$ is the tubular neighborhood of $f(S^1)$ in $S^1 \times S^2$. Let V be $S^1 \times S^2 - \text{Int}B^3 - \text{Int}N(f(S^1))$. Note that $\partial V = S^2 \amalg T^2$.

(2) There is a non-divisible cycle $\zeta \in H_1(\partial V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ associated with a cycle $\in \iota(H_1(V; \mathbb{Z}))$ such that $\zeta \notin \iota(H_1(V; \mathbb{Z}))$. We have what is written in the fifth line of Proof of Theorem 4.11.

Let F and G be oriented closed connected surface $\subset S^4$. An (F, G) -link is a 2-dimensional closed oriented submanifold $L = (J, K) \subset S^4$ such that J (resp. K) is diffeomorphic to F (resp. G). Let $L = (J, K)$ and $L' = (J', K')$ be (F, G) -links in S^4 . We say that L and L' are *surface-link-cobordant* if there is an embedding map $f : (F \amalg G) \times [0, 1] \hookrightarrow S^4 \times [0, 1]$ with the following properties:

For $t = 0, 1$, $f((F \amalg G) \times [0, 1]) \cap (S^4 \times \{t\})$ is $f((F \amalg G) \times \{t\})$.

$f((F \amalg G) \times \{0\})$ in $S^4 \times \{0\}$ is L .

$f((F \amalg G) \times \{1\})$ in $S^4 \times \{1\}$ is L' .

[24, §2] proved that if two surface-links are surface-link-cobordant and the alinking number of one of the two is zero, then the alinking number of the other is zero. We generalize it and prove the following:

Proposition 7.10. *Let $L = (J, K)$ and $L' = (J', K')$ be (F, G) -links in S^4 . Suppose that L and L' are surface-link-cobordant. Then we have the following:*

$$\text{alk}(J \subset L, K \subset L) = \text{alk}(J' \subset L', K' \subset L').$$

$$\text{alk}(K \subset L, J \subset L) = \text{alk}(K' \subset L', J' \subset L').$$

Proof of Proposition 7.10. Take a compact oriented 4-manifold P such that $\partial P = f(F \times [0, 1]) \cup (\text{a Seifert hypersurface for } J) \cup (\text{a Seifert hypersurface for } J')$ (resp. $f(G \times [0, 1]) \cup (\text{a Seifert hypersurface for } K) \cup (\text{a Seifert hypersurface for } K'))$. Consider $P \cap f(G \times [0, 1])$ (resp. $P \cap f(F \times [0, 1])$) and Definition 4.9. \square

Proof of Theorem 4.12. In Proof of Theorem 4.11, since $H_1(V, \partial V; \mathbb{Z})$ has a nontrivial torsion in general, σ_1 is $\iota(\beta)$ or a non-divisible 1-cycle associated with $\iota(\beta)$. That is, k in Proof of Theorem 4.11 is not ± 1 in general. Now, since $H_1(V, \partial V; \mathbb{Z})$ is torsion-free,

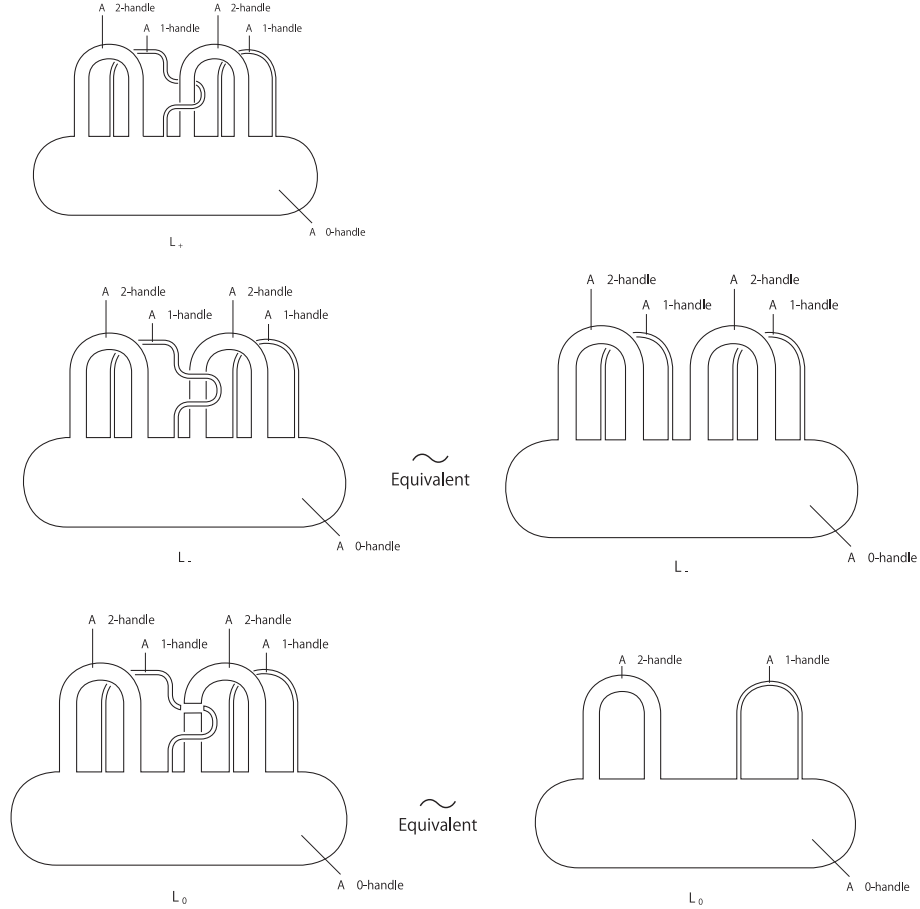


Figure 7.2: A (1,2)-pass-move-triple (L_+, L_-, L_0)

$\iota(\beta) = \pm\sigma_1$ and $|\text{lk}(\beta, \tau_1)| = |\text{lk}(\sigma_1, \tau_1)|$. Hence the alinking number is $|\text{lk}(\sigma_1, \tau_1)|$. Hence the alinking number is the pseudo-alinking number.

By Theorem 4.11, Theorem 4.12 holds. \square

Proof of Corollary 4.13. Use the isotopy which changes [19, Figure 4.4] into [19, Figure 4.3] and vice versa. Use 4-dimensinal 1-handles. We obtain a Seifert hypersurface for L whose homology groups are torsion-free. \square

Proof of Theorem 4.14. There is a (1,2)-pass-move-triple (L_+, L_-, L_0) with the following properties (see Figure 7.2): L_+, L_- are diffeomorphic to S^2 . L_0 is diffeomorphic to $S^2 \amalg T^2$. A seifert hypersurface for L_+ (resp, L_-) is diffeomorphic to $(S^2 \times S^1) \sharp (S^2 \times S^1) - \text{Int}B^3$, where \sharp denotes the connected-sum. Note that it has a handle decomposition

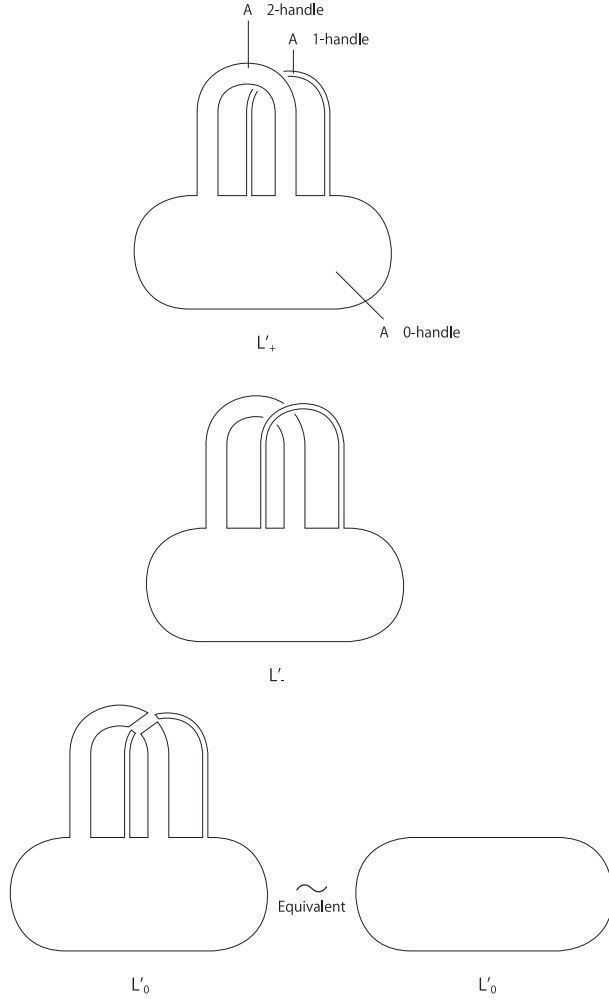


Figure 7.3: A (1,2)-pass-move-triple (L'_+, L'_-, L'_0)

(a 3-dimensional 0-handle) \cup (two 3-dimensional 1-handles) \cup (two 3-dimensional 2-handles). A seifert hypersurface for L_0 is diffeomorphic to $(S^2 \times D^1) \natural (D^2 \times S^1)$, where \natural denotes the boundary-connected-sum. Note that it has a handle decomposition (a 3-dimensional 0-handle) \cup (a 3-dimensional 1-handle) \cup (a 3-dimensional 2-handle).

A 1-Alexander matrix for L_+ (resp, L_- , L_0) is $\begin{pmatrix} t & t-1 \\ 0 & t \end{pmatrix}$ (resp. $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$, (0)).

There is a (1,2)-pass-move-triple (L'_+, L'_-, L'_0) with the following properties (see Figure 7.3): L'_+ , L'_- , and L'_0 are diffeomorphic to S^2 . A seifert hypersurface for L_+ (resp, L_-) is diffeomorphic to $(S^2 \times S^1) - \text{Int} B^3$. Note that it has a handle decomposition (a 3-dimensional 0-handle) \cup (a 3-dimensional 1-handle) \cup (a 3-dimensional 2-handle).

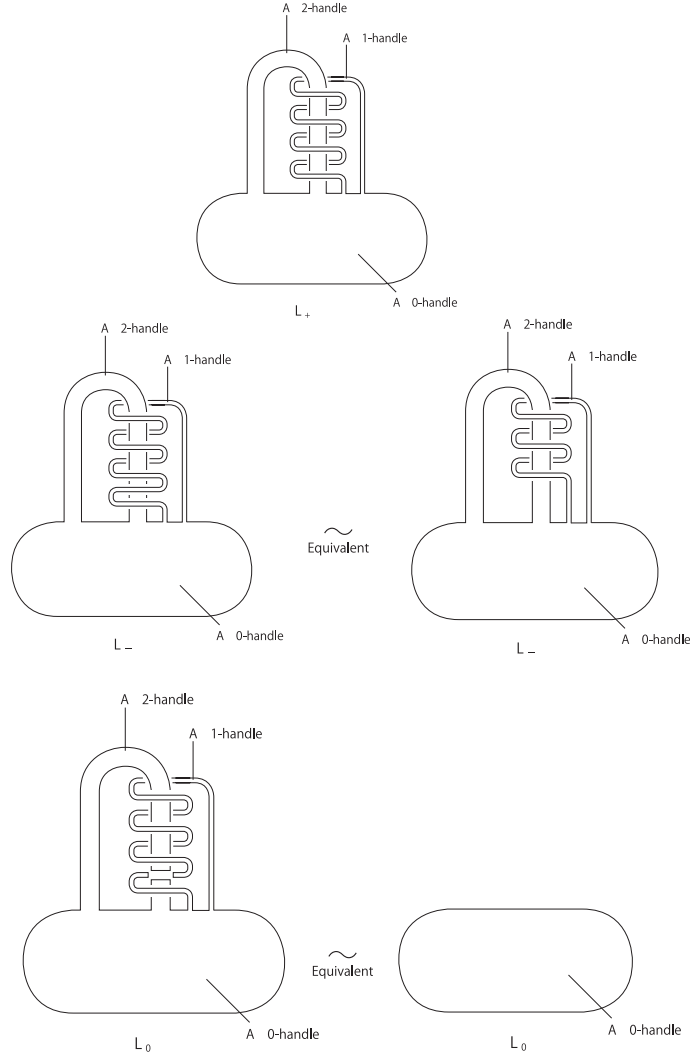


Figure 7.4: A (1,2)-pass-move-triple (L_+, L_-, L_0)

A seifert hypersurface for L'_0 is diffeomorphic to a 3-ball, which can be regarded a 3-dimensional 0-handle. Note that L'_0 is a trivial 2-knot. A 1-Alexander matrix for L'_+ (resp, L'_- , L'_0) is (t) (resp. (1) , ϕ), where ϕ denotes the empty matrix.

By these example, Theorem 4.14 holds. \square

Proof of Proposition 4.3. Propositions 4.3 follows from Proposition 4.5. \square

Proof of Proposition 4.5. There are the following examples (see Figures 7.4 and 7.5).

Let L_+ (resp. L_- , L'_+ , L'_-) be an (S^2, T^2) -link and bound a Seifert hypersurface which

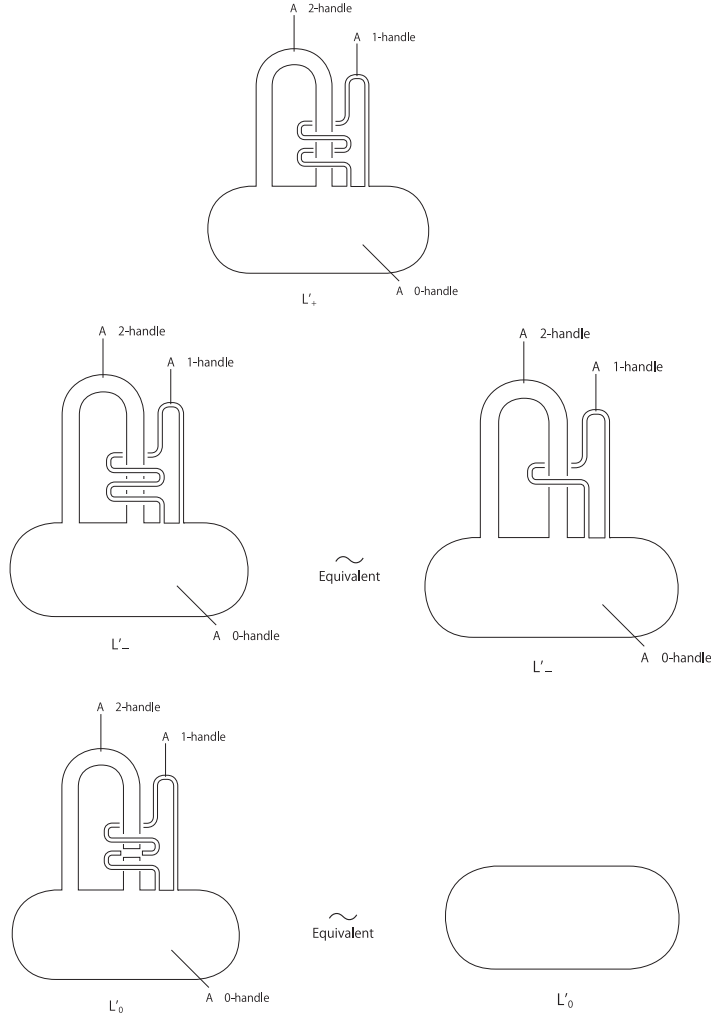


Figure 7.5: A (1,2)-pass-move-triple (L'_+, L'_-, L'_0)

is diffeomorphic to $(S^2 \times D^1) \natural (D^2 \times S^1)$. Note that it has a handle decomposition (a 3-dimensional 0-handle) \cup (a 3-dimensional 1-handle) \cup (a 3-dimensional 2-handle) and that it is not diffeomorphic to $(S^2 \times S^1) - \text{Int} B^3$. We can assume that a Seifert matrix of L_+ (resp. L_- , L'_+ , L'_-) is a 1×1 -matrix (4) (resp. (3), (2), (1)). We can suppose that L_0 (resp. L'_0) is a trivial 2-knot such that (L_+, L_-, L_0) (resp. (L'_+, L'_-, L'_0)) is a (1, 2)-pass-move-triple. \square

8. PROOF OF RESULTS IN §6

Proof of Theorem 6.2. Lemmas 8.1 and 8.2 imply Theorem 6.2.

Lemma 8.1. *Theorem 6.2 is true if K is homotopy type equivalent to S^{4k+1} .*

Proof of Lemma 8.1. Recall that $S_{2k+1}(V) - N_{2k+1}(V)$ is represented by the intersection product on $H_{2k+1}(V; \mathbb{Z})$. By the Poincaré duality $\det(S_{2k+1}(V) - N_{2k+1}(V)) = 1$. Hence $\hat{\Delta}_K(1) = 1$.

By the Poincaré duality a $2k$ -Alexander matrix induces an injective homomorphism on $H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q})$. Hence the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\det(t \cdot S_{2k+1}(V) - N_{2k+1}(V))$ is the $(2k+1)$ - $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial by Proposition 3.2. Hence the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\det(t \cdot S_{2k+1}(V) - N_{2k+1}(V))$ is topological invariant of K . Hence the $\mathbb{Q}[t, t^{-1}]$ -balanced class of $\det(t \cdot S_{2k+1}(V) - N_{2k+1}(V))$ does not depend on the choice of V .

Let V' be a Seifert hypersurface for K . Let $S_{2k+1}(V')$ be a positive $(2k+1)$ -Seifert matrix for K and $N_{2k+1}(V')$ its related negative Seifert matrix. Let $\hat{\Delta}'_K(t) = \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V') - t^{\frac{-1}{2}} \cdot N_{2k+1}(V'))$. Recall $\hat{\Delta}_K(t) = \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V))$. It suffices to prove that $\hat{\Delta}'_K(t) = \hat{\Delta}_K(t)$.

Since V (resp. V') is $(4k+2)$ -dimensional and ∂V (resp. $\partial V'$) is PL homeomorphic to the standard sphere, $\text{rank} H_{2k+1}(V; \mathbb{Z})$ (resp. $\text{rank} H_{2k+1}(V'; \mathbb{Z})$) is even. Therefore there is an integer n such that $\hat{\Delta}'_K(t) = t^n \cdot \hat{\Delta}_K(t) \cdots (*)$ holds. (Note that $\hat{\Delta}_K(1) = \hat{\Delta}'_K(1) = 1$.)

By Propositions 3.3 and 3.4, $N_{2k+1}(V) = {}^t S_{2k+1}(V)$. By the Poincaré duality the number of the rows of $S_{2k+1}(V)$ and that of the columns of it are the (same) even nonnegative integer. Hence we have the following: Let $M(t) = \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V))$. Then $M(t) = {}^t M(t^{-1})$. Hence $\hat{\Delta}_K(t)$ (resp. $\hat{\Delta}'_K(t)$) has a form

$$\sum_{\rho=0}^{\rho=l} a_\rho \cdot t^{\frac{\rho}{2}} + \sum_{\rho=0}^{\rho=l} a_\rho \cdot t^{-\frac{\rho}{2}}.$$

By this fact and the above identity $(*)$, we have $\hat{\Delta}'_K(t) = \hat{\Delta}_K(t)$.

This completes the proof of Lemma 8.1. □

Lemma 8.2. *Theorem 6.2 is true if K is homotopy type equivalent to $S^{2k+1} \times S^{2k}$.*

Proof of Lemma 8.2. Let V be a Seifert hypersurface for K . There are two cases (see Note (2) to Definition 6.1):

(I) Any $2k$ -Alexander matrix associated with V induces an injective map on $H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q})$.

(II) No $2k$ -Alexander matrix associated with V induces an injective map on $H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q})$.

Lemmas 8.3 and 8.4 imply Lemma 8.2.

Lemma 8.3. *Lemma 8.2 is true in the case (II).*

Proof of Lemma 8.3. Since V satisfies (II), the normalized Alexander polynomial of K defined by using V is zero by Definition 6.1. By Definition 3.1 the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial of K defined by using V is the $\mathbb{Q}[t, t^{-1}]$ -balanced class of zero.

Let V' be another Seifert hypersurface for K . By Definition 3.1 the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial of K defined by using V' is the $\mathbb{Q}[t, t^{-1}]$ -balanced class of zero even if V' satisfies (I) not (II). Therefore, by Definitions 3.1 and 6.1 and Proposition 3.2 the normalized Alexander polynomial of K defined by using V' is zero.

This completes the proof of Lemma 8.3. \square

Lemma 8.4. *Lemma 8.2 is true in the case (I).*

Proof of Lemma 8.4. In the same manner as written in the first part of Proof of Theorem 4.1 it suffices to prove the following case: V and V' are Seifert hypersurfaces for K . V' is obtained from V by a surgery by using an embedded $(4k + 3)$ -dimensional i -handle h^i ($1 \leq i \leq 4k + 2$). This surgery may change a $(2k + 1)$ -Alexander matrix associated with V for K only if $i = 2k + 1, 2k + 2$.

Lemma 8.3 implies that if V' satisfies (II), Lemma 8.4 holds. Hence it suffices to prove the case where V' satisfies (I).

Hence both V and V' satisfy (II). The dual handle of h^{2k+2} is a $(4k + 3)$ -dimensional $(2k + 1)$ -handle. Hence it suffices to prove the $i = 2k + 2$ case. Call the core of the attaching part of h^{2k+2} , C .

There are two cases:

- (1) $[C] \in H_{2k+1}(V, \mathbb{Z})$ is order finite.
- (2) $[C] \in H_{2k+1}(V, \mathbb{Z})$ is order infinite.

The case (2) is divided into two cases:

- (2-1) $[C] \in H_{2k+1}(V, \mathbb{Z})$ is order infinite. For all $(2k + 1)$ -cycle α , the intersection product $[C] \cdot \alpha = 0$.
- (2-2) $[C] \in H_{2k+1}(V, \mathbb{Z})$ is order infinite. There is a $(2k + 1)$ -cycle α such that the intersection product $[C] \cdot \alpha$ is nonzero.

Lemmas 8.5, 8.6, and 8.7 imply Lemma 8.4.

Lemma 8.5. *Lemma 8.4 holds in the case (1).*

Proof of Lemma 8.5. This surgery does not change a $(2k + 1)$ -Alexander matrix associated with V for K . This completes the proof of Lemma 8.5. \square

Lemma 8.6. *Lemma 8.4 holds in the case (2-1).*

Proof of Lemma 8.6. There is an Alexander matrix associated with V which has a row (or column) all of whose elements are zero. Hence the $\mathbb{Q}[t, t^{-1}]$ -Alexander polynomial of K is the $\mathbb{Q}[t, t^{-1}]$ -balanced class of zero. By Definitions 3.1 and 6.1 the normalized Alexander polynomial is zero.

This completes the proof of Lemma 8.6. \square

Lemma 8.7. *Lemma 8.4 holds in the case (2-2).*

Proof of Lemma 8.7. Let \bar{h}^{2k+1} be the dual handle of the $(4k+3)$ -dimensional $(2k+2)$ -handle h^{2k+2} . Let C' be the core of the attaching part of \bar{h}^{2k+1} . Note that $(V \times [0, 1]) \cup h^{2k+2}$ is diffeomorphic to $(V' \times [0, 1]) \cup \bar{h}^{2k+1}$.

Consider the exact sequence by a pair $((V \times [0, 1]) \cup h^{2k+2}, V)$, where we regard V as $V \times \{0\}$:

$$\cdots \rightarrow H_*(V; \mathbb{Z}) \rightarrow H_*(V \times [0, 1] \cup h^{2k+2}; \mathbb{Z}) \rightarrow H_*(V \times [0, 1] \cup h^{2k+2}, V; \mathbb{Z}) \rightarrow \cdots$$

and the exact sequence by a pair $((V' \times [0, 1]) \cup \bar{h}^{2k+1}, V')$, where we regard V' as $V' \times \{0\}$:

$$\cdots \rightarrow H_*(V'; \mathbb{Z}) \rightarrow H_*(V' \times [0, 1] \cup \bar{h}^{2k+1}; \mathbb{Z}) \rightarrow H_*(V' \times [0, 1] \cup \bar{h}^{2k+1}, V'; \mathbb{Z}) \rightarrow \cdots$$

By the existence of the $(2k+1)$ -cycle α , $[C'] \in H_{2k}(V'; \mathbb{Z})$ is order finite.

Let $\xi \in H_{2k+1}(V; \mathbb{Z})$ be a non-divisible element associated with $[C]$. Since $H_{2k}(\partial V; \mathbb{Z})$ is torsion-free and the intersection product $\xi \cdot \alpha \neq 0$, there is a $(2k+1)$ -cycle $\eta \in H_{2k+1}(V; \mathbb{Z})$ such that $\eta \cdot \xi = 1$. We can suppose that η is a non-divisible element associated with α . Therefore we have the following: $A(t)$ (resp. $A'(t)$) is $\det(t^{\frac{1}{2}} \cdot S_{2k+1}(V) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V))$ (resp. $\det(t^{\frac{1}{2}} \cdot S_{2k+1}(V') - t^{\frac{-1}{2}} \cdot N_{2k+1}(V'))$) an $(2k+1)$ -Alexander matrix associated with V (resp. V'). We have

$$A'(t) = \left(\begin{array}{cc|ccc} 0 & t^{\frac{1}{2}} & 0 & \cdot & \cdot & 0 \\ -t^{\frac{-1}{2}} & 0 & a_{23} & \cdot & \cdot & a_{2n} \\ \hline 0 & a_{23} & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & a_{2n} & & & & \end{array} \right) A(t)$$

or

$$A'(t) = \left(\begin{array}{cc|ccc} 0 & -t^{\frac{1}{2}} & 0 & \cdot & \cdot & 0 \\ t^{\frac{-1}{2}} & 0 & a_{23} & \cdot & \cdot & a_{2n} \\ \hline 0 & a_{23} & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & a_{2n} & & & & \end{array} \right) A(t)$$

This completes the proof of Lemma 8.7. \square

This completes the proof of Lemma 8.4. □

This completes the proof of Lemma 8.2. □

This completes the proof of Theorem 6.2. □

Proof of Theorem 6.3. In [21, Theorem 3.3] we proved the following: There is a Seifert hypersurface V_* for K_* ($*$ = +, −, 0) with an associated $(2k + 1)$ -Seifert matrix $S_{2k+1}(V_*)$ and its related $(2k + 1)$ -negative Seifert matrix $N_{2k+1}(V_*)$ with the following properties:

(i)

$$\begin{aligned} & \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V_+) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V_+)) \\ & - \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V_-) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V_-)) \\ & = (t - 1) \cdot \det(t^{\frac{1}{2}} \cdot S_{2k+1}(V_0) - t^{\frac{-1}{2}} \cdot N_{2k+1}(V_0)). \end{aligned}$$

(ii) $S_{2k+1}(V_+)$ and $S_{2k+1}(V_-)$ are $2\nu \times 2\nu$ -matrices ($\nu \in \mathbb{N}$). $S_{2k+1}(V_+)$ is a $(2\nu - 1 \times 2\nu - 1)$ -matrix.

(iii) The $2k$ -Alexander matrix associated with each Seifert hypersurface defines an injective map on $H_{2k}(\Pi_{-\infty}^\infty V \times [-1, 1]; \mathbb{Q}) \rightarrow H_{2k}(\Pi_{-\infty}^\infty Y; \mathbb{Q})$.

Hence

$$\hat{\Delta}_{K_+}(t) - \hat{\Delta}_{K_-}(t) = (t^{\frac{1}{2}} - t^{\frac{-1}{2}}) \cdot \hat{\Delta}_{K_0}(t).$$

This completes the proof of Theorem 6.3. □

Proof of Proposition 6.6. In the same manner as written in the first part of Proof of Theorem 4.1 it suffices to prove the following case: V and V' are Seifert hypersurfaces for K . V' is obtained from V by a surgery by using an embedded $(4k + 3)$ -dimensional i -handle h^i ($1 \leq i \leq 4k + 2$). The pseudo-twinkling number may change only if $i = 2k + 1, 2k + 2$.

The dual handle of h^{2k+2} is a $(4k + 3)$ -dimensional $(2k + 1)$ -handle. Therefore it suffices to prove the following two cases under the condition $i = 2k + 2$.

(1) There is a non-divisible $(2k + 1)$ -cycle $\tau \subset V$ such that for any $(2k + 1)$ -cycle $\alpha \subset V$ the intersection product $\tau \cdot \alpha$ in V is zero

(2) There is not such a cycle as in (1).

By Poincaré duality and Mayor-Vietoris exact sequence, τ is a non-divisible cycle in V associated with $* \times S^{2k+1}$ in $K = \partial V$.

The above two cases (1) and (2) are divide into four cases.

(1-1) V satisfies (1). V' satisfies the condition made from (1) by replacing V with V' in (1).

(1-2) V satisfies (1). V' satisfies the condition made from (2) by replacing V with V' in (2).

(2-1) V satisfies (2). V' satisfies the condition made from (1) by replacing V with V' in (1).

(2-2) V satisfies (2). V' satisfies the condition made from (2) by replacing V with V' in (2).

Lemmas 8.8, 8.9, 8.11, and 8.12 imply Proposition 6.6.

Lemma 8.8. *Proposition 6.6 holds in the case (2-2).*

Proof of Lemma 8.8. By Definition 6.5 the pseudo-twinkling number defined by using V (resp. V') is zero. \square

Lemma 8.9. *Proposition 6.6 holds in the case (2-1).*

Proof of Lemma 8.9. When we obtain V' from V by using h^{2k+2} , there does not appear τ' in V' as in Definition 6.1. The case (2-1) does not occur. \square

Lemmas 8.8 and 8.9 and their proof imply Claim 8.10.

Claim 8.10. *The pseudo-twinkling number is zero in the case (2).*

Lemma 8.11. *Proposition 6.6 holds in the case (1-2).*

Proof of Lemma 8.11. By Definition 6.5 the pseudo-twinkling number defined by using V' is zero.

Let C be the attaching part of h^{2k+2} . Note that the Seifert pairing $s(C, C) = 0$.

Under the condition (1-2), τ must be a non-divisible $(2k+1)$ -cycle associated with $[C]$. Hence $s(\tau', \tau') = 0$. \square

Lemma 8.12. *Proposition 6.6 holds in the case (1-1).*

Proof of Lemma 8.12. The pseudo-twinkling number defined by using V is $s(\tau, \tau)$. Let C be the core of the attaching part of h^{2k+2} . There are two cases.

(i) τ is a non-divisible cycle in V associated with C .

(ii) Else.

In the case (i), V and V' satisfy (1-2) not (1-1).

The cases (ii) follows from Theorem 6.7 and its proof as written below because its proof does not depend of the choice of V .

This completes the proof of Lemma 8.12. \square

This completes the proof of Proposition 6.6. \square

Proof of Theorem 6.7. Let V be a Seifert hypersurface for K .

In the case (2) of Proof of Proposition 6.6. By Claim 8.10 the pseudo-twinkling number is zero. In this case no $2k$ -Alexander matrix associated with V induces an injective map.

By Definition 6.1 the normalized Alexander polynomial is zero. Hence Theorem 6.7 holds in this case.

In the case (1) of Proof of Proposition 6.6. Let $\{\alpha_1, \dots, \alpha_\nu\}$ be a basis of $H_{2k+1}(V; \mathbb{Z})/\text{Tor}$. We can suppose that $\alpha_1 = \tau$. Then $t^{\frac{1}{2}}S - t^{-\frac{1}{2}}tS$ is written as follows, where $a_{\#}$ is an integer and a_{11} is the pseudo-twinkling number.

$$\begin{pmatrix} \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{11}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} & \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{12}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} & \cdot & \cdot & \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{1*}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} \\ \cdot & & & & \\ \cdot & & & & \\ \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{*1}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} & & & & \end{pmatrix} Q(t)$$

$$= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \begin{pmatrix} \frac{a_{11}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} & a_{12} & \cdot & \cdot & a_{1*} \\ \cdot & & & & \\ \cdot & & & & \\ \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{*1}}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot a_{21}} & & & & \end{pmatrix} Q(t)$$

Hence $\frac{t^{\frac{1}{2}}S - t^{-\frac{1}{2}}tS}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \Big|_{t=1}$ is written as follows.

$$\begin{pmatrix} \frac{a_{11}}{0} & a_{12} & \cdot & \cdot & a_{1*} \\ 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & & & \end{pmatrix} Q(1)$$

Since $Q(1)$ is a nonsingular matrix and its determinant is $+1$, $\frac{\hat{\Delta}_K(t)}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \Big|_{t=1} = a_{11}$.

Note 8.13. If we define V as in (1), $\iota : H_{2k+1}(\partial V; \mathbb{Z}) \rightarrow H_{2k+1}(V; \mathbb{Z})$ has the property in (2).

(1) Take $S^{2k+1} \times S^{2k+1}$. Let f be an embedding map $S^{2k+1} \hookrightarrow S^{2k+1} \times S^{2k+1}$. Let $f(S^{2k+1}) \subset S^{2k+1} \times S^{2k+1}$. Suppose that the induced map $f_* : H_{2k+1}(S^{2k+1}; \mathbb{Z}) \rightarrow H_{2k+1}(S^{2k+1} \times S^{2k+1}; \mathbb{Z})$ is $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ with $1 \mapsto (n, 0)$, where $|n| > 1$. Here, we fix a generator of $H_{2k+1}(S^{2k+1}; \mathbb{Z})$ and that of $H_{2k+1}(S^{2k+1} \times S^{2k+1}; \mathbb{Z})$. Let V be $(S^{2k+1} \times S^{2k+1}) - \text{Int}N(f(S^{2k+1}))$, where $N(f(S^{2k+1}))$ is the tubular neighborhood of $f(S^{2k+1})$ in $S^{2k+1} \times S^{2k+1}$.

(2) Let g be a generator of $H_{2k+1}(\partial V; \mathbb{Z}) \cong \mathbb{Z}$. Then $\rho(g)$ is a divisible cycle $\in H_{2k+1}(V; \mathbb{Z})$.

9. A PROBLEM

Problem 9.1. (1) If an invariant of 2-dimensional oriented closed submanifold $\subset S^4$ satisfies the identity in Theorem 4.4 associated with the $(1, 2)$ -pass-move, then is it essentially the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial?

(2) If an invariant of $(4k + 1)$ -dimensional submanifolds $\subset S^{4k+3}$ whose homotopy type is S^{4k+1} or $S^{2k} \times S^{2k+1}$ satisfies the identity in Theorem 6.3 as written there, then is it essentially the normalized Alexander polynomial (resp. the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial)?

Note. If the answer to Problem 9.1.(1) is positive, it is a new characterization of the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial of 2-dimensional closed oriented submanifolds in S^4 . If the answer is negative, we may encounter a new invariant.

If the answer to Problem 9.1.(2) is positive, it is a new characterization of the normalized Alexander polynomial (resp. the $\mathbb{Z}[t, t^{-1}]$ -Alexander polynomial) of high dimensional knots in the case. If the answer is negative, we may encounter a new invariant.

There arise similar problems on the local-move-identities in [11, 21] to Problem 9.1.

REFERENCES

- [1] J. Adams: On the nonexistence of elements of Hopf invariant one, *Ann. of Math. (2)*, 72, 20-104, 1960.
- [2] J.W.Alexander: Topological invariants of knots and links, *Trans. Amer. Math. Soc.*, 30 (1928) 275-306.
- [3] J. Conway: An enumeration of knots and links and some of their related properties, *In: Computational problems in Abstract Algebra (Oxford 1967) Welsh Pergamon Press* 329-358, 1970.
- [4] A. Haefliger: Differentiable imbeddings, *Bull. Amer. Math. Soc.* 67 (1961) 109-112.
- [5] A. Haefliger: Plongements différentiables de variétés dans variétés. *Comment. Math. Helv.* 36 (1962) 47 - 82.
- [6] J. Hoste: The first coefficient of the Conway polynomial *Proc. Amer. Math. Soc.* 95 (1985) 299-302.
- [7] I. M. James and J. H. C. Whitehead: The homotopy theory of sphere bundles over spheres. I, *Proc. Lond. Math. Soc.* 4, 196-218, 1954.
- [8] V. F. R. Jones: Hecke Algebra representations of braid groups and link polynomials, *Ann. of Math. (2)* 126, 335-388, 1987.
- [9] L. H. Kauffman: On Knots, *Ann. of Math. Stud.* 115 (1987).
- [10] Louis H. Kauffman, State models and the Jones polynomial, *Topology* 26 (1987) 395-407.
- [11] L. H. Kauffman and E. Ogasa: Local moves of knots and products of knots, *Volume three of Knots in Poland III, Banach Center Publications* 103 (2014), 159-209, arXiv: 1210.4667 [math.GT].
- [12] L. H. Kauffman and E. Ogasa: Local moves of knots and products of knots II, arXiv: 1406.5573 [math.GT].
- [13] L. H. Kauffman and E. Ogasa: Brieskorn submanifolds, Local moves on knots, and knot products, arXiv: 1504.01229 [math.GT].

- [14] R. Kirby: The topology of 4-manifolds, *Lecture Notes in Math.* (SpringerVerlag) 1374 (1989).
- [15] J. Levine: Polynomial invariant of knots of codimension two, *Ann. of Math. (2)* 84, (1966) 537-554.
- [16] J. Levine: An algebraic classification of some knots of codimension two, *Comment. Math. Helv.* 45, (1970) 185-198.
- [17] E. Ogasa: Intersectional pairs of n -knots, local moves of n -knots and invariants of n -knots, *Math. Res. Lett.* 5 (1998) 577-582, Univ. of Tokyo preprint UTMS 95-50.
- [18] E. Ogasa: The intersection of spheres in a sphere and a new geometric meaning of the Arf invariants, *J. Knot Theory Ramifications* 11 (2002) 1211-1231, Univ. of Tokyo preprint series UTMS 95-7, arXiv: 0003089 [math.GT].
- [19] E. Ogasa: Ribbon-moves of 2-links preserve the μ -invariant of 2-links, *J. Knot Theory Ramifications* 13 (2004) 669-687, UTMS 97-35, arXiv: 0004008 [math.GT].
- [20] E. Ogasa: Ribbon-moves of 2-knots: The Farber-Levine pairing and the Atiyah-Patodi-Singer-Casson-Gordon-Ruberman $\tilde{\eta}$ invariant of 2-knots, *Journal of Knot Theory and Its Ramifications* 16 (2007) 523-543, arXiv: 0004007 [math.GT], UTMS 00-22, arXiv: 0407164 [math.GT].
- [21] E. Ogasa: Local move identities for the Alexander polynomials of high-dimensional knots and inertia groups, *J. Knot Theory Ramifications* 18 (2009) 531-545, UTMS 97-63, arXiv: 0512168 [math.GT].
- [22] E. Ogasa: A new obstruction for ribbon-moves of 2-knots: 2-knots fibred by the punctured 3-torus and 2-knots bounded by the Poincaré sphere, arXiv: 1003.2473 [math.GT].
- [23] E. Ogasa: An introduction to high dimensional knots, arXiv: 1304.6053 [math.GT].
- [24] N. Sato: Cobordisms of semi-boundary links *Topology Appl.* 18 (1984) 225-234.
- [25] H. Whitney: Differentiable manifolds, *Ann. of Math. (2)* 37 (1936) 645-680.
- [26] H. Whitney: The Self-intersections of a smooth n -manifold in $2n$ -space, *Ann. of Math. (2)* 45 (1944) 220-246.
- [27] W. T. Wu: On the isotopy of C^r -manifolds of dimension n in Euclidean $(2n + 1)$ -space, *Sci. Record (N.S.)* 2 (1958) 271-275.

Eiji Ogasa Computer Science, Meijigakuin University, Yokohama, Kanagawa, 244-8539, Japan pqr100pqr100@yahoo.co.jp ogasa@mail1.meijigakuin.ac.jp